PART A

Derivation of general three dimensional conduction equations in Cartesian coordinate

Generally the heat conduction problem consists of finding the temperature at any time and at any point within a specified solid that has been heated to a known initial temperature distribution and whose surface has been subjected to a known set of boundary conditions.

Consider a solid as shown in Fig 2-4 with heat conducting in and out of a unit volume in all three coordinate directions x, y and z

Making energy balance (1)

--- [1]

$$q_x + q_y + q_z + q_{gen} = q_{x+dx} + q_{y+dy} + q_{z+dz} + \frac{dE}{dt}$$

where,

$$q_x = -kdydz\frac{\partial T}{\partial x}$$

$$q_{x+dx} = -\left[k\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx\right]dydz$$

$$q_y = -kdxdz\frac{\partial T}{\partial y}$$

$$q_{y+dy} = -\left[k\frac{\partial T}{\partial y} + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)dy\right]dxdz$$

$$q_z = -kdxdy\frac{\partial T}{\partial z}$$

$$q_{z+dz} = -\left[k\frac{\partial T}{\partial z} + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)dz\right]dxdy$$

$$q_{gen} = \dot{q}\,dxdydz$$

$$\frac{dE}{dt} = \rho c dxdydz \frac{\partial T}{\partial t}$$

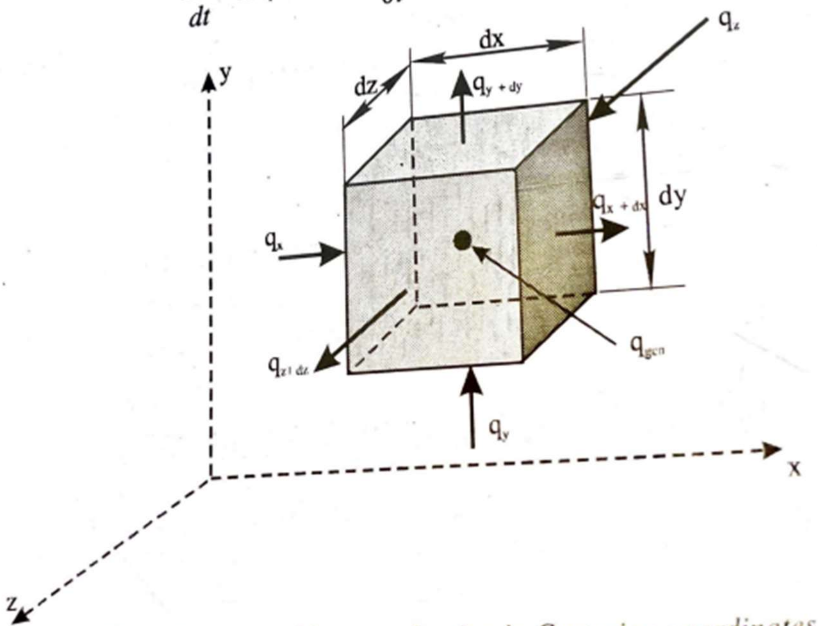


Fig 2-4 : Three dimensional heat conduction in Cartesian coordinates

Substituting all the values in equations [1] above general three dimensional heat conduction equation becomes

$$-kdydz\frac{\partial T}{\partial x} - kdxdz\frac{\partial T}{\partial y} - kdxdy\frac{\partial T}{\partial z} + \dot{q}\,dxdydz$$

$$= -\left[k\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx\right]dydz$$

$$-\left[k\frac{\partial T}{\partial y} + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)dy\right]dxdz$$

$$-\left[k\frac{\partial T}{\partial z} + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)dz\right]dxdy + \rho c dxdydz\frac{\partial T}{\partial t}$$

Rearranging and simplifying the above equation

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c\frac{\partial T}{\partial t}$$

If thermal conductivity k is constant, the above equation becomes

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{\rho C}{k}\frac{\partial T}{\partial t}$$

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}} \qquad \text{--- [2]}$$

In the above equation the quantity α is known as thermal diffusivity of the material. Rate of heat diffusion through the material is faster if α is higher. The term ρc is known as thermal heat capacity.

Higher value of α may be either due to higher value of thermal conductivity or lower value of thermal heat capacity. Lower value of thermal heat capacity means the energy moving through the material. would be absorbed to a lesser degree and used to raise the temperature of the material. This means more energy is available for further transfer

Discussion on 3-D conduction in cylindrical and spherical coordinate systems (No derivation).

Cylindrical Coordinates

Cylindrical Coordinates are expressed in radius (r), axis (z) and longitude (ϕ) as shown in fig

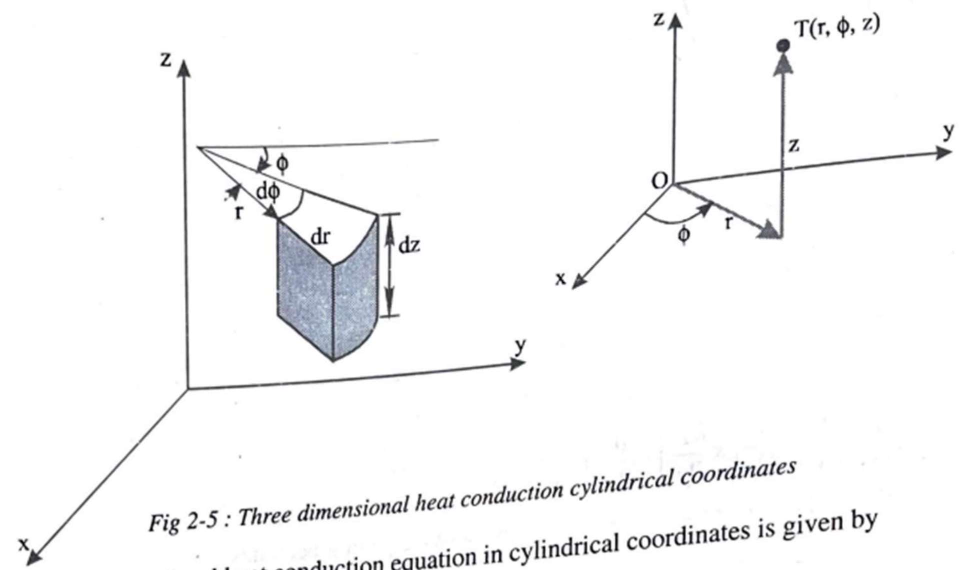


Fig 2-5 : Three dimensional heat conduction cylindrical coordinates

Three dimensional heat conduction equation in cylindrical coordinates is given by

$$\boxed{\frac{\partial^2 T}{\partial r^2}+\frac{1\partial T}{r\partial r}+\frac{1\partial^2 T}{r^2\partial\phi^2}+\frac{\partial^2 T}{\partial^2 z^2}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}} \quad \text{--- [3]}$$

Spherical Coordinates

Spherical coordinates system expressed in (r, ϕ, z) is shown in Fig 2-6.

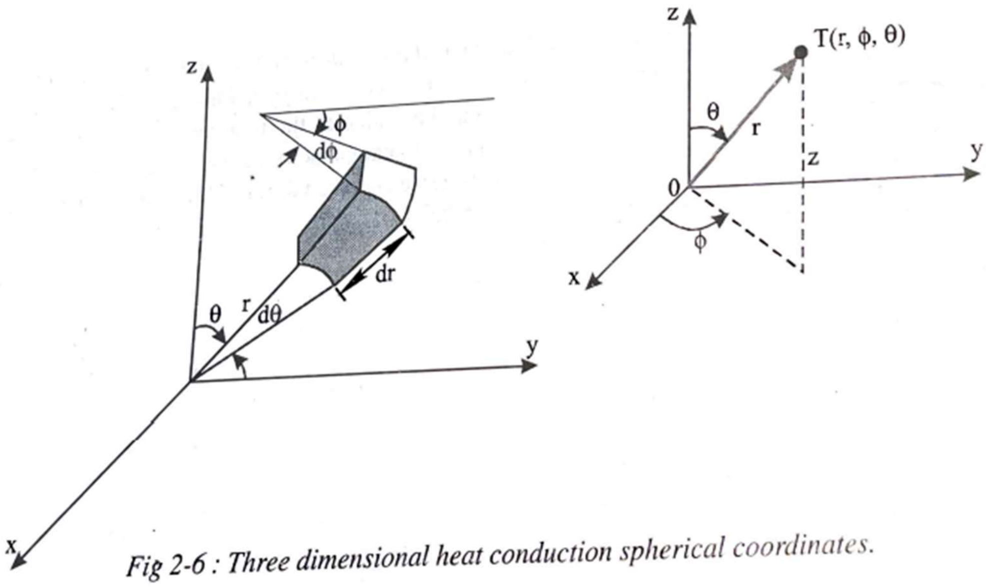


Fig 2-6 : Three dimensional heat conduction spherical coordinates.

Three dimensional heat conduction equation in spherical coordinates is given by,

$$\boxed{\frac{1}{r}\frac{\partial^2}{\partial r^2}(rT)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial T}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\theta^2}+\frac{\dot{q}}{k} = \frac{1\partial T}{\alpha\partial t}} \quad \text{--- [4]}$$

SPECIAL FORMS OF HEAT CONDUCTION EQUATION

From equation (3) of section ■ 2.3 some special cases of particular interest are as follows.

1. Laplace equation Considering the three dimensional heat conduction equation in Cartesian Co-ordinates, we have

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \text{--- (1)}$$

Using Laplacian operator ∇^2, the above equation becomes,

$$\nabla^2 T+\frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \text{--- (2)}$$

$$\text{Where}\quad \nabla^2 T = \frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}$$

If heat generation is absent and the process is steady flow, $\dot{q} = 0$ and,

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2} = 0$$

Under these conditions, equation (2) reduces to

$$\nabla^2 T = 0 \quad \text{--- [3]}$$

The above equation is known as Laplace equation.

2. Poisson's equation In many cases, the temperature at any point in a material doesn' t change with time,

$$\text{i.e.,} \quad \frac{\partial T}{\partial t} = 0.$$

From equation (2),

$$\nabla^2 T + \frac{\dot{q}}{k} = 0 \qquad \text{--- [4]}$$

The above equation is known as **Poisson equation.**

3. Fourier equation

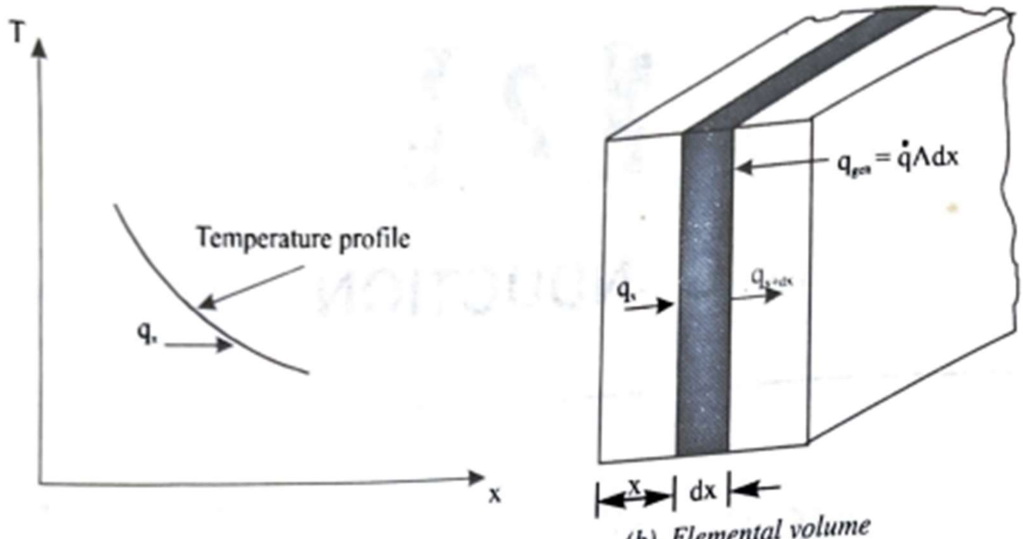
For unsteady state heat transfer with no internal heat generation then equation (2) above reduces to

$$\nabla^2 T = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{--- [5]}$$

One-dimensional conduction equations in rectangular, cylindrical and spherical coordinates for plane and composite walls.

RECTANGULAR OR CARTESIAN CO-ORDINATES

Consider a one dimensional system as shown in Fig 2-1. In the steady state system, the temperature doesn't change with time. If the temperature changes with time the system is known as unsteady state system. This is the general case where the temperature is not constant



(a) Variation of temperature *(b) Elemental volume*

Fig 2-1 : One dimensional heat conduction

(qgen) (q x+dex)

Fig 2.1 : One dimensional ...

Let,

$$q_x = \text{Energy conducted in LHS of the element} = -kA\frac{\partial T}{\partial x}$$

$$q_{gen} = \text{Energy generated within the elemental strip}$$

$$= \dot{q}\, A dx$$

$$q_{x+dx} = \text{Energy conducted out of the RHS of the element}$$

$$= -kA\left.\frac{\partial T}{\partial x}\right|_{x+dx}$$

$$\frac{dE}{dt} = \text{Change in internal energy}$$

$$= \rho c A\frac{\partial T}{\partial t}$$

where

ρ = density

c = specific heat of material

$\dot{q}$ = energy generated per unit volume

Making energy balance for an elemental strip dx,

$$q_x + q_{gen} = \frac{dE}{dt} + q_{x+dx}$$

$$\text{i.e.,}\quad -kA\frac{\partial T}{\partial x} + \dot{q}\, A dx = \rho c A\frac{\partial T}{\partial t}dx - kA\left.\frac{\partial T}{\partial x}\right|_{x+dx}$$

$$= \rho c A\frac{\partial T}{\partial t}dx - A\left[kA\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx\right]$$

Writing in differential form

$$\boxed{\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \dot{q} = \rho C\frac{\partial T}{\partial t}} \qquad \text{--- [1]}$$

The above equation is known as one dimensional heat conduction equation.

CYLINDRICAL CO-Ordinate's

The Cartesian coordinate system discussed above is not applicable to determine heat conduction in cylinders, cones, spheres etc. When heat conduction takes place through such geometries, cylindrical co-ordinate systems are used, since co-ordinate surfaces coincide with the boundary surfaces of the region. . For heat transfer analysis, consider an infinitesimal cylindrical volume element shown in figure 2.2.

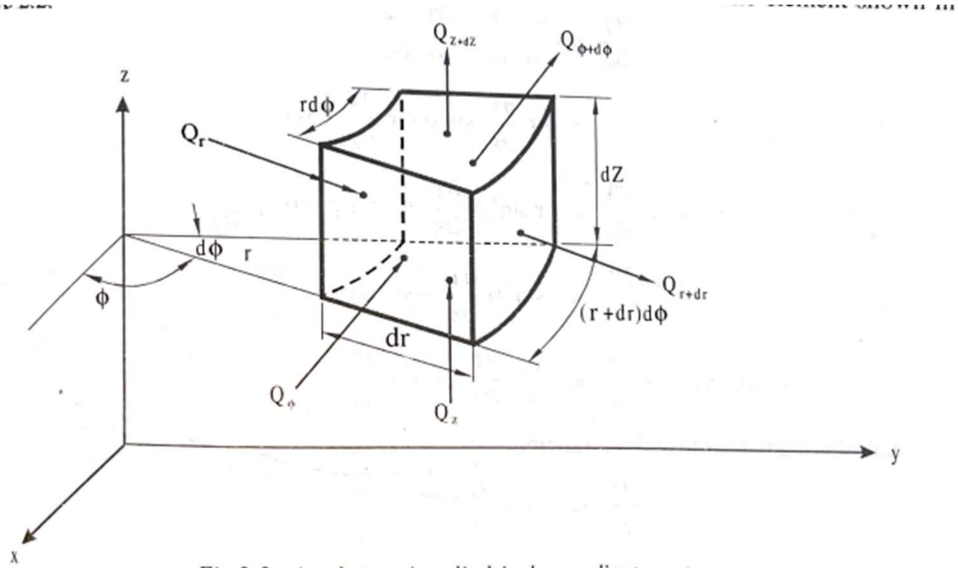


Fig 2-2 : An element in cylindrical co-ordinate system

The following assumptions are made while deriving the heat conduction equation

- Thermal conductivity k, density ρ and specific heat C for the material do not change with position
- Heat generation rate is uniform per unit volume per unit time

From the figure 2.2,

$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$\phi = \tan^{-1}(y/x)$$
$$\text{and } z = z.$$

Using chain rule and partially differentiating T with reference to r we have,

$$\frac{\partial T}{\partial r} = \frac{\partial T}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial T}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \cos\phi \,.\, \frac{\partial T}{\partial x} + \sin\phi\frac{\partial T}{\partial y}$$

$$\therefore \cos\phi\frac{\partial T}{\partial r} = \cos^2\phi\frac{\partial T}{\partial x} + \sin\phi\,.\,\cos\phi\frac{\partial T}{\partial y} \qquad \text{--- [1]}$$

Similarly partially differentiating T with reference to φ we have,

$$\frac{\partial T}{\partial \phi} = \frac{\partial T}{\partial x}\cdot\frac{\partial x}{\partial \phi} + \frac{\partial T}{\partial y}\cdot\frac{\partial y}{\partial \phi}$$

$$= -r\frac{\partial T}{\partial x}\sin\phi + r\frac{\partial T}{\partial y}\cos\phi$$

$$\therefore \quad \sin\phi \frac{\partial T}{\partial \phi} = -r\sin^2\phi \frac{\partial T}{\partial x} + r\sin\phi \, . \cos\phi \frac{\partial T}{\partial y}$$

or
$$\frac{\sin\phi}{r}\frac{\partial T}{\partial \phi} = -\sin^2\phi \frac{\partial T}{\partial x} + \sin\phi\cos\phi \frac{\partial T}{\partial y} \qquad \text{--- [2]}$$

$$\therefore \quad \sin\phi . \cos\phi \frac{\partial T}{\partial y} = \frac{\sin\phi}{r}\frac{\partial T}{\partial \phi} + \sin^2\phi \frac{\partial T}{\partial x} \qquad \text{--- [3]}$$

Substituting (3) in (1) and rearranging,

$$\frac{\partial T}{\partial x} = \cos\phi \frac{\partial T}{\partial r} - \frac{\sin\phi}{r}\frac{\partial T}{\partial \phi} \qquad \text{--- [4]}$$

$$\frac{\partial T}{\partial y} = \sin\phi \frac{\partial T}{\partial r} + \frac{\cos\phi}{r}\frac{\partial T}{\partial \phi} \qquad \text{--- [5]}$$

From (4) and (5),

$$\frac{\partial^2 T}{\partial x^2} = \cos\phi \frac{\partial T}{\partial r}\left(\frac{\partial T}{\partial x}\right) - \frac{\sin\phi}{r}\frac{\partial T}{\partial \phi}\left(\frac{\partial T}{\partial x}\right)$$

$$= \cos\phi \frac{\partial T}{\partial r}\left[\cos\phi \frac{\partial T}{\partial r} - \frac{\sin\phi}{r}\frac{\partial T}{\partial \phi}\right] - \frac{\sin\phi}{r}\frac{\partial T}{\partial \phi}\left[\cos\phi \frac{\partial T}{\partial r} - \frac{\sin\phi}{r}\frac{\partial T}{\partial \phi}\right]$$

$$\frac{\partial^2 T}{\partial x^2} = \cos^2\phi \frac{\partial^2 T}{\partial r^2} + \frac{\cos\phi . \sin\phi}{r^2}\frac{\partial T}{\partial \phi} + \frac{\sin^2\phi}{r}\frac{\partial T}{\partial r}$$
$$+ \frac{\sin^2\phi}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\cos\phi . \sin\phi}{r^2} \cdot \frac{\partial T}{\partial \phi} \qquad \text{--- [6]}$$

and
$$\frac{\partial^2 T}{\partial y^2} = \cos\phi \frac{\partial T}{\partial r}\left(\frac{\partial T}{\partial y}\right) - \frac{\sin\phi}{r} \cdot \frac{\partial T}{\partial \phi}\left(\frac{\partial T}{\partial y}\right)$$

$$= \sin^2\phi \frac{\partial^2 T}{\partial r^2} + \frac{\cos^2\phi}{r}\frac{\partial T}{\partial r} - \frac{\cos\phi . \sin\phi}{r^2} \cdot \frac{\partial T}{\partial \phi} + \frac{\cos\phi}{r^2}\frac{\partial^2 T}{\partial \phi^2}$$
$$- \frac{\cos\phi . \sin\phi}{r^2}\frac{\partial T}{\partial \phi} \qquad \text{--- [7]}$$

Adding (6) and (7) we have

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}$$

or
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} \qquad \text{--- [8]}$$

If the heat conduction is unidirectional i.e., along the radial direction r only, the above equation reduces to

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \quad \text{--- (9)}$$

But from equation (1) we have for constant thermal conductivity k,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{k} = \frac{\rho C}{k}\frac{\partial T}{\partial t} \quad \text{--- (10)}$$

Substituting (9) in (10) above,

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\dot{q}}{k} = \frac{\rho C}{k}\frac{\partial T}{\partial t}} \quad \text{--- (11)}$$

SPHERICAL CO-ORDINATES

Consider an infinitesimal spherical element of volume dV shown in fig. Considering heat conduction only along the direction r, we can derive heat conduction equation in a single co-ordinate

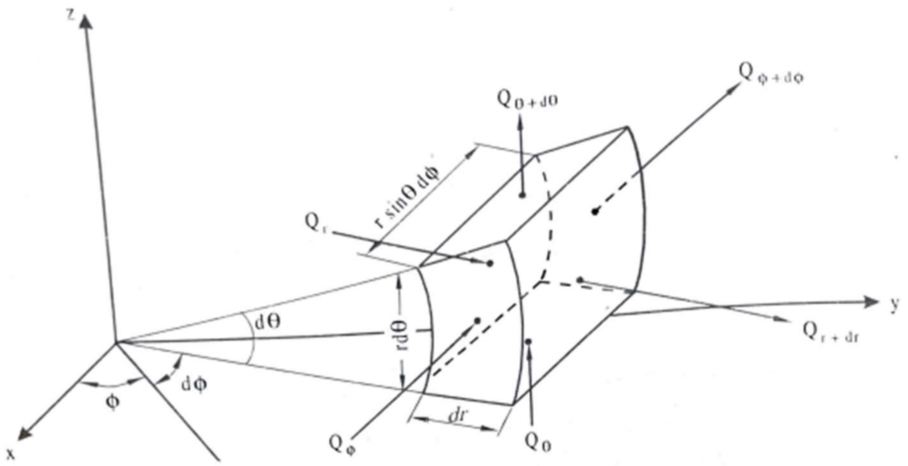


Fig 2-3 : A spherical co-ordinate system

Considering $\theta - \phi$ plane; r-direction

$$\text{Heat in, } Q_r = -k\,(r\,d\theta \sin\theta\, d\phi)\,\frac{\partial T}{\partial r}\,dt \quad \text{--- [1]}$$

$$\text{Heat out, } Q_{r+dr} = Q_r + \frac{\partial}{\partial r}(Q_r)\,dr \quad \text{--- [2]}$$

Heat storage in the elemental volume due to heat storage in x-direction,

$$dQ_r = Q_r - Q_{r+dr}$$

$$= -\frac{\partial}{\partial r}(Q_r)\,dr$$

$$= -\frac{\partial}{\partial r}\left[-k(rd\theta \sin\theta d\phi)\frac{\partial T}{\partial r}\dot{d}t\right]dr$$

$$= k\,d\theta \sin\theta\, d\,\theta\, dr\,\frac{\partial}{\partial r}\left[r^2\frac{\partial T}{\partial r}\right]dt$$

$$= k\,(dr.\,rd\theta.r\sin\theta.d\phi)\,\frac{1}{r^2}\frac{\partial}{dr}\left[r^2\frac{\partial T}{\partial r}\right]dt$$

$$= k.dV\,\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)dt \qquad \text{--- [3]}$$

Heat generated within the control volume,

$$= \dot{q}\;dV\,dt \qquad \text{--- [4]}$$

Rate of change of energy within the control volume

$$= \rho C\,dV\,\frac{\partial T}{\partial t}.dt \qquad \text{--- [5]}$$

For energy balance we have,

Total heat storage in control volume + internal heat generation

= Rate of change of energy within the control volume.

From above equations (3), (4) and (5) we have,

$$kdV\,\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)dt + \dot{q}\,dVdt = \rho\,Cd\dot{V}\,\frac{\partial T}{\partial t}dt$$

or

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{\dot{q}}{k} = \rho\,C\,\frac{\partial T}{dt}} \qquad \text{--- [6]}$$

The above equation is one dimensional heat conduction equation in spherical coordinate along radial direction.

General equation for one dimensionil heat conduction T

The one dimensional heat conduction equation in the Cartesian (rectangular), cylindrical, and spherical coordinate systems is given by a single general equation as

$$\boxed{\frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n k\frac{\partial T}{\partial r}\right) + \dot{q} = \rho c\frac{\partial T}{\partial r}} \qquad \text{--- [7]}$$

Where n = 0 for rectangular coordinates

1 for cylindrical coordinates

2 for spherical coordinates.

Overall heat transfer coefficient.

In many instances it is customary to express the heat flow rate in the cases of single or multi-layered plane walls and cylinders with convection at the boundaries in terms of an overall conductance or overall heat transfer coefficient U.

A. PLANE WALL Consider a plane wall exposed to a hot fluid A on one side and a cold fluid B on the other side. The heat transfer is expressed as

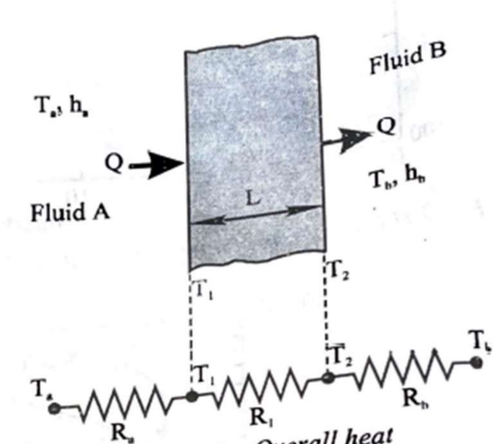


Fig. 3.12 : Overall heat transfer coefficient through a plane wall

$$Q = h_a A (T_a - T_1) = \frac{kA}{L}(T_1 - T_2) = h_b A (T_2 - T_b)$$

i.e., $$Q = \frac{(T_a - T_1)}{\frac{1}{h_a A}} = \frac{(T_1 - T_2)}{\frac{L}{kA}} = \frac{(T_2 - T_b)}{\frac{1}{h_b A}}$$

Adding the numerators and denominators of the above equation,

$$Q = \frac{T_a - T_b}{\frac{1}{h_a A} + \frac{L}{kA} + \frac{1}{h_b A}} \quad \text{--- [1]}$$

i.e., $$Q = \frac{T_a - T_b}{R_a + R_1 + R_b} \quad \text{--- [2]}$$

The overall heat transfer coefficient due to combined heat transfer by convection and conduction is given as,

$$Q = UA\,\Delta T_{overall}$$

$$= \frac{\Delta T_{overall}}{\frac{1}{UA}} \qquad \text{--- [3]}$$

Comparing equations [1] and [2] with [3],

$$\frac{1}{U} = \frac{1}{h_a} + \frac{L}{k} + \frac{1}{h_b} = R_a + R_1 + R_b$$

Rearranging,

$$\boxed{U = \frac{1}{\left(\frac{1}{h_a}\right) + \frac{L}{k} + \frac{1}{h_b}} = \frac{1}{R_a + R_I + R_b}} \qquad \text{--- [4]}$$

B. HOLLOW CYLINDER

Consider a hollow cylindrical tube with a hot fluid A flowing inside it and a cold fluid B flowing outside its surface. Let T_a, T_b be the corresponding temperatures and h_u, h_b be the corresponding heat transfer coefficients. The arrangement with an equivalent electric circuit is shown in Fig. 3-13

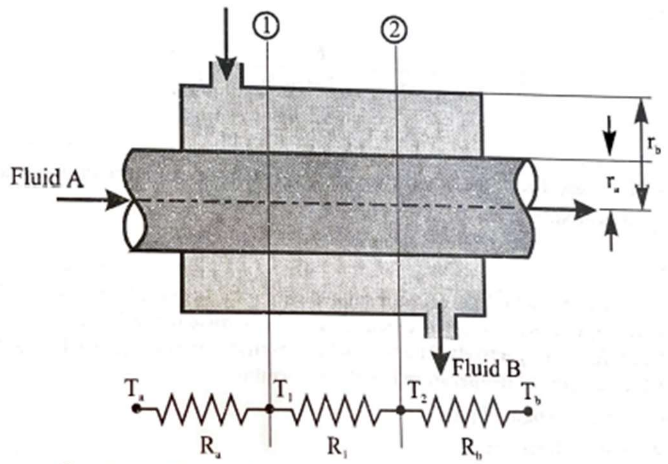


Fig. 3.13 : Overall heat transfer coefficient through a cylinder

The heat flow rate is given by,

$$Q = \frac{T_a - T_b}{\frac{1}{h_a A_a} + \frac{\ln(r_b/r_a)}{2\pi k L} + \frac{1}{h_b A_b}} \qquad \text{--- [5]}$$

The above equation can be written based on the inside area A_a and outside area A_b of the tube or cylinder.

$$Q = \frac{T_a - T_b}{\left(\frac{1}{h_a} + \frac{A_a \ln(r_b/r_a)}{2\pi k L} + \frac{A_a}{A_b}\frac{1}{h_b}\right)\frac{1}{A_a}} \quad \text{--- [6]}$$

or

$$Q = \frac{T_a - T_b}{\left(\frac{A_b}{A_a}\frac{1}{h_a} + \frac{A_a \ln(r_b/r_a)}{2\pi k L} + \frac{1}{h_b}\right)\frac{1}{A_b}} \quad \text{--- [7]}$$

The overall heat transfer coefficient due to combined conduction and convection is given as,

$$Q = \frac{\Delta T_{overall}}{\frac{1}{U_a A_a}} = \frac{\Delta T_{overall}}{\frac{1}{U_b A_b}} \quad \text{--- [8]}$$

Comparing equations [6] and [7] with [8],

$$U_a = \frac{1}{\frac{1}{h_a} + \frac{A_a \ln(r_b/r_a)}{2\pi kL} + \frac{A_a}{A_b}\frac{1}{h_b}}$$

$$U_b = \frac{1}{\frac{A_b}{A_a}\frac{1}{h_a} + \frac{A_b \ln(r_b/r_a)}{2\pi kL} + \frac{1}{h_b}} \quad \text{--- [9]}$$

where V_a and V_b are the inside and outside overall heat transfer coefficients based on the respective inside and outside areas of the cylinder or tube.

Thermal contact resistance

Consider two solid bars brought into contact as shown in Fig. 3-10. The sides of the bars are insulated so that heat flows only in axial direction. The temperature profile through the solids experiences a sudden drop across the interface between the two materials. This temperature drop at the contact plane between the two materials is due to thermal contact resistance.

Consider the enlarged view of the interface as shown in Fig. 3-10.The direct contact between the

solids takes place only between a few spots whereas the gap between the solids is either filled with air or surrounding fluid. Since radiation effects are negligible at normal temperature and since there can not be any convection in such a thin layer of the fluid, heat transfer through the fluids filling the gaps or voids takes place mainly by conduction. Thus two principal contribution to the heat transfer at the contact surface are

1. The solid to solid conduction at the point of contact

2. The conduction through fluids filling the gaps or voids created by contact

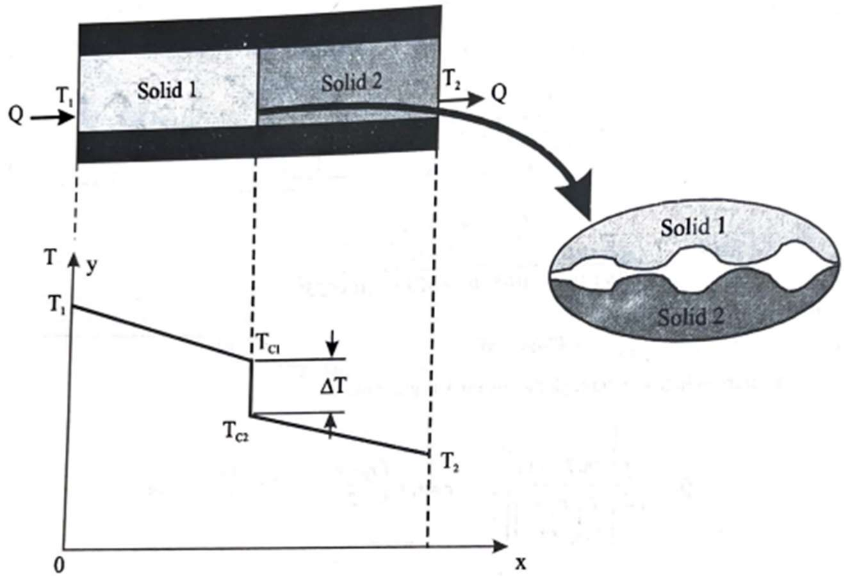
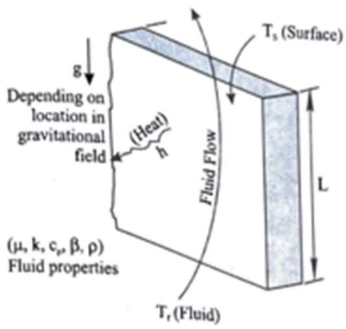


Fig. 3.10 : Temperature drop across a contact resistance

Part B

Free or Natural Convection: Application of dimensional analysis for free convection-

When the heat transfer takes place by actual motion of the molecules without external assistance then heat transfer by convection is known as free convection



The fluid velocity in case of free convection depends upon the following parameters;

1. Temperature difference between solid surface and bulk fluid, ΔT
2. Acceleration due to gravity, g
3. Coefficient of volumetric expansion of fluid, β

The change in the volume when temperature changes can be expressed as

$$dV = V_1 \beta (T_2 - T_1)$$

where

dV - change in volume (m^3) $= V_2 - V_1$

β = Coefficient of volumetric expansion of fluid, (m^3/m^3 °C)

T_2 - Final temperature (°C)

T_1 - Initial temperature (°C)

Therefore , free convection heat transfer coefficient is a function of following variables

Variable	Symbol	Dimensions
Fluid density	ρ	ML^{-3}
Dynamic viscosity	μ	$ML^{-1}T^{-1}$
Thermal conductivity	k	$MLT^{-3}\theta^{-1}$
Specific heat	C_p	$L^2 T^{-2}\theta^{-1}$
Characteristic length	D	L
Temperature difference	ΔT	θ

Therefore, convective heat transfer coefficient is expressed as

$$h = f(\rho, \mu, k, Cp, D, \Delta T, \beta, g)$$

However, in free convection, (ΔT β g) will be treated as single parameter as the velocity of fluid particles is a function of these parameters. Therefore, equation (i) can be expressed as

$$f(h, \rho, \mu, k, Cp, D, (\Delta T \beta g)) = 0$$

Convective heat transfer coefficient, h is dependent variable and remaining are independent variables.

Total number of variables, n = 7

Number of fundamental units, m = 4

According to Buckingham's π-theorem, number of π-terms is given by the difference of total number of variables and number of fundamental units.

Number of π-terms = (n-m) = 7-4 = 3

These non-dimensional π-terms control the forced convection phenomenon and are expressed as,

$f(\pi_1, \pi_2, \pi_3) = 0....(i)$

Each π_1 -term is expressed as:

$\pi_1 = \mu^a k^b \rho^c D^d h..(ii)$

Writing down each term in above equation in terms of fundamental dimensions

$M^0 L^0 T^0 \theta^0 = (ML^{-1}T^{-1})^a (MLT^{-3}\theta^{-1})^b (ML^{-3})^c (L)^d MT^{-3}\theta^{-1}$

Comparing the powers of M, we get

$0 = a + b + c + 1,$

$a + b + c = -1$

Comparing powers of L, we get

$0 = -a + b + c + d$

Comparing powers of T, we get

$0 = -a - 3b - c - 3$

Comparing powers of θ, we get

$b = -1$

Substituting the values of 'a', 'b', 'c' and 'd' in equation (ii), we get

$\pi_1 = \mu^0 k^{-1} \rho^0 D^1 h$

$\pi_1 = hD/K$

The second π_2–term is expressed as

$\pi_2 = \mu^a k^b \rho^c D^d C_P$

After following same steps we get

$\pi_2 = \mu C_p / K = Pr$

The third π_3 –term is expressed as

$\pi_3 = \mu^a k^b \rho^c D^d ((\Delta T \beta g)$

After following same steps, we get

$$\pi_3 = D^3(\Delta T\beta g)/v^2$$

Substituting the values of π_1, π_2, π_3 in equation (i), we get

$$f(\frac{hD}{K}, \frac{\mu C_P}{K}, \frac{D^3(\Delta T\beta g)}{v^2}) = 0$$

$$\frac{hD}{K} = \varphi(\frac{\mu C_P}{K}, \frac{D^3(\Delta T\beta g)}{v^2})$$

$Nu = \varphi(Pr, Gr)$ as

$$Gr = D^3(\Delta T\beta g)/v^2$$

The above correlation is generally expressed as,

$$Nu = C(Pr)^a(Gr)^b$$

The constant C and exponents a and b are determined through experiments.

physical significance or Grashoff number;

8. Grashoff Number

Grashoff number is defined as the ratio of product of inertia force and buoyance force to the square of viscous force.

$$Gr = \frac{Inertia\ force \times Buoyance\ force}{(Viscous\ force)^2}$$

$$\boxed{Gr = \frac{\rho V^2 \times \rho\beta g\Delta TL^3}{(\mu V)^2} = \frac{\rho^2\beta g\Delta TL^3}{\mu^2}} \quad \text{--- [9]}$$

where V is the velocity of the fluid caused by buoyance force ($\beta g\ \Delta T$).

use of correlations of free convection in vertical, horizontal and inclined flat plates, vertical and horizontal cylinders and spheres,

VERTICAL PLATE

1. Uniform Wall Temperature

For constant wall temperature *McAdams* correlated the average *Nusselt number* with following expression.

$$\boxed{Nu_m = c\,(Gr_L Pr)^n = cRa_L^{\,n}} \qquad \text{--- [1]}$$

where

L = The vertical height of the plate

Gr = Grashoff number

$$= \frac{\beta g L^3 (T_W - T_\infty)}{\nu^2}$$

Nu_m = Nusselt number

$$= \frac{h_m L}{k}$$

$Ra_L = Gr_L Pr$

The constant c and exponent n are given in table 7-1.

Table 7-1 : Constant c and exponent n

Type of flow	Range of $Gr_L\,Pr$	c	n
Laminar	10^4 to 10^9	0.59	$\frac{1}{4}$
Turbulent	10^9 to 10^{13}	0.10	$\frac{1}{3}$

two more equations are proposed by Churchill1 and other for laminar flow for all values of Prandtl Number

$$\boxed{Nu_m = 0.68 + \frac{0.67 Ra_L^{1/4}}{\left[1 + \left(0.492/Pr\right)^{9/16}\right]^{4/9}}} \quad \text{for } 10^{-1} < Ra_L < 10^9 \text{ --- [2]}$$

For both laminar and turbulent flow

$$\boxed{Nu_m = 0.825 + \frac{0.387 Ra^{1/6}}{\left[1 + \left(0.492/Pr\right)^{9/16}\right]^{8/27}}} \quad \text{for } 10^{-1} < Ra_L < 10^{12} \text{ --- [3]}$$

In all the above equations the physical properties are evaluated at, $T_f = \dfrac{(T_W + T_\infty)}{2}$

2. Uniform Wall Heat Flux

The following correlations are proposed for the local Nusselt number under uniform wall heat flux .

For laminar flow

$$\boxed{Nu_x = 0.60(Gr_x^* Pr)^{1/5}} \quad \text{for } 10^5 < Gr_x^* Pr < 10^{11} \qquad \text{--- [4]}$$

For turbulent flow

$$\boxed{Nu_x = 0.568\,(Gr_x^* Pr)^{0.22}} \quad \text{for } 2 \times 10^{13} < Gr_x^* Pr < 10^{16} \qquad \text{--- [5]}$$

where Gr_x^* = Modified Grashof number

$$= Gr_x\, Nu_x = \frac{\beta g (T_W - T_\infty) x^3}{\nu^2} \times \frac{q_W x}{(T_W - T_\infty)}$$

$$= \frac{\beta g q_W x^4}{k \nu^2}$$

q_W = Constant wall heat flux

Nu_x = Local Nusselt number

$$= \frac{x h_x}{k}$$

The average *Nusselt number* for equations [4] and [5] are given by

$$Nu_m = 1.25\,[Nu_x]_{x=L} \qquad \text{for } 10^5 < Gr_x^* Pr < 10^{11}$$

$$Nu_m = 1.136\,[Nu_x]_{x=L} \qquad \text{for } 2 \times 10^{13} < Gr_x^* Pr < 10^{16}$$

Equation [2] suggested by *Churchil and Chu* also applies for uniform heat flux conditions. It can be expressed in terms of modified *Grashof number* Gr^* by substituting $Ra_L = Gr_L Pr$ and $Gr_L^* = Gr_L Nu_m$. Thus for laminar flow,

$$\boxed{Nu_m^{1/4}(Nu_m - 0.68) = \frac{0.67(Gr_L^* Pr)^{1/4}}{\left[1 + \left(0.492/Pr\right)^{9/16}\right]^{4/9}}} \qquad \text{--- [6]}$$

HORIZONTAL PLATE

The average Nusselt number for free convection on a horizontal plate depends on whether the plate surface is warmer or cooler than the surrounding fluid and whether the surface is facing up or down.

1. Uniform Wall Temperature

The empirical relation given by *McAdams* is expressed as,

$$Nu_m = c\,(Gr.Pr)^n \qquad \text{--- [1]}$$

The constant c and exponent n are listed in table 7-2

Table 7-2 : Constant c and exponent n for equation [1]

Plate position or Orientation	$Gr_L Pr$	c	n	Flow regime
Hot surface facing up or Cold surface facing down	10^5 to 2×10^7	0.54	$1/4$	Laminar
	2×10^7 to 3×10^{10}	0.14	$1/3$	Turbulent
Hot surface facing down or Cold surface facing up	3×10^5 to 3×10^{10}	0.27	$1/4$	Laminar

In the above equation [1]

$$Nu_m = \frac{h_m L}{k} \quad \text{and} \quad Gr_L = \frac{\beta g (T_W - T_\infty) L^3}{\nu^2}$$

The characteristic length L of the plate is given by

$$L = \frac{Surface\,area\ Plate}{Perimeter}$$

For square plate, L = Length of a side

For rectangular plate, L = Arithmetic mean of two dimensions

For circular disk L = 0.9 times the diameter

2. Uniform Wall Heat flux

For a horizontal plate with the **heated surface facing upward**

$$\boxed{Nu_m = 0.13 (Gr_L Pr)^{1/3}} \rightarrow \text{for } Gr_L Pr < 2 \times 10^8 \quad \text{--- [2]}$$

$$\boxed{Nu_m = 0.16 (Gr_L Pr)^{1/3}} \rightarrow \text{for } 5 \times 10^8 < Gr_L Pr < 10^{11} \quad \text{--- [2]}$$

For the horizontal plate with the *heated surface facing downward*,

$$\boxed{Nu_m = 0.58 (Gr_L Pr)^{1/5}} \quad \text{for } 10^6 < Gr_L Pr < 10^{11} \quad \text{--- [3]}$$

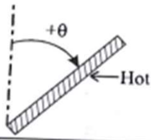
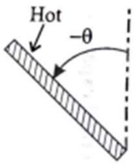
The physical properties in equation [2] and [3] are evaluated at *a mean temperature*.

$$T_m = T_W - 0.25 (T_W - T_\infty)$$

The thermal expansion coefficient (β) is evaluated at $(T_W + T_\infty)/2$.

7-4-2 <u>INCLINED PLATE :</u> For an inclined plate following table 7.3 gives some useful relations.

Table 7-3

<table>
<tr><th>FLOW CONDITION / REGIME</th><th>RELATION</th><th>REMARKS</th><th>NOTATIONS</th></tr>
<tr>
<td>1. Inclined plate with uniform heat flux, heated surface facing downward

</td>
<td>$Nu_m = 0.56(Gr_L \, Pr \cos\theta)^{1/4}$
Note :
1. All physical properties are evaluated at mean temperature $T_m = T_w - 0.25\,(T_w - T_\infty)$ and β is evaluated at $T_\infty + 0.25\,(T_w - T_\infty)$
2. Angle θ is considered positive if the hot surface is facing down.</td>
<td>Average Nusselt number for free convection from an inclined plate subject to approximately uniform wall flux, valid for $+\theta < 88^\circ$, and $10^5 < Gr_L \, Pr < 10^{11}$</td>
<td>θ = Angle which the surface makes with the vertical.</td>
</tr>
<tr>
<td>2. Inclined plate with uniform heat flux, heated surface facing upward.

</td>
<td>$Nu_m = 0.145\left[(Gr_L\,Pr)^{1/3} - (Gr_C\,Pr)^{1/3}\right] + 0.56(Gr_C \, Pr \cos\theta)^{1/4}$
Where, Gr_c is the transition Grashof number dependent on angle θ as given in the table below.
<table><tr><th>Degrees</th><th>Gr_c</th></tr><tr><td>− 15</td><td>5×10^9</td></tr><tr><td>− 30</td><td>10^9</td></tr><tr><td>− 60</td><td>10^8</td></tr><tr><td>− 75</td><td>10^6</td></tr></table>Note :
1. All physical properties are evaluated at mean temperature $T_m = T_w - 0.25\,(T_w - T_\infty)$ and β is evaluated at $T_\infty + 0.25\,(T_w - T_\infty)$
2. Angle θ is considered negative if the hot surface is facing up.</td>
<td>Average Nusselt number for free convection from an inclined plate subjected to approximately uniform wall flux valid for $Gr_L Pr < 10^{11}$, $Gr_L > Gr_c$ and $-15^\circ < \theta < -75^\circ$</td>
<td>Gr_c = Transition Grashof number which depends on θ
θ = Angle of inclination of the surface with the vertical.</td>
</tr>
</table>

* All physical properties are evaluated at film temperature, $T_f = 0.5\,(T_w + T_\infty)$ unless otherwise stated.

VERTICAL CYLiNDER, If the thickness of the thermal boundary layer is much smaller than the cylinder radius, then the average Nusselt number for free convection on a vertical cylinder is same as that of a vertical plate.

Hence McAdams correlation holds good here also i.e.,

$$\boxed{Nu_m = c\,(Gr_L\,Pr)^n = c\,Ra_L^{\,n}} \quad \text{---[1]}$$

where the values of c and n are given in table 7-3. In the above case the length L of the plate corresponds to the height of the cylinder.

For fluids having *Prandtl number* equal to 0.7 and greater than 0.7, the vertical cylinder may be treated as a vertical flat plate when

$$\frac{L/D}{(Gr_L)^{1/4}} < 0.025 \text{ where } D \text{ is the cylinder diameter.}$$

When the vertical cylinder is subjected to uniform wall heat flux, the local Nusselt numbers are given by the same empirical relations used for a vertical plate.

7-4-6 HORIZONTAL CYLINDER

For an isothermal horizontal cylinder, *Churchill* and *Chu* have proposed the following relation,

$$\boxed{Nu_m^{1/2} = 0.60 + \frac{0.387\,Ra_D^{\,1/6}}{\left[1+\left(0.559/Pr\right)^{9/16}\right]^{8/27}}} \quad \text{for } 10^{-4} < Ra_D < 10^{12} \text{ --- [2]}$$

where $$Nu_m = \frac{hD}{k};\; Ra_D = Gr_D Pr = \left(\frac{\beta g (T_W - T_\infty) D^3}{\nu^2}\right) Pr$$

Morgan presented the following relation from the horizontal isothermal cylinder,

$$\boxed{Nu_m = \frac{hD}{k} = c\,Ra_D^{\,n}} \quad \text{for } 10^{-10} < Ra_D < 10^{12} \quad \text{--- [3]}$$

The values of constant c and exponent n are listed in table 7-3.

Table 7-3

Ra_D	c	n
$10^{-10} - 10^{-2}$	0.675	0.058
$10^{-2} - 10^{2}$	1.02	0.148
$10^{2} - 10^{4}$	0.850	0.188
$10^{4} - 10^{7}$	0.480	0.250
$10^{7} - 10^{12}$	0.125	0.333