# Example Sums to Determine Pay off matrix

1. A and B are playing a game with 3 coins 25p, 50p, 100p according to the following rules

i) If the sum is even A will win B's coin

(ii) If the sum is odd, B will win A's coin

Formulate the pay off matrix

→

$$
\begin{array}{cc|ccc|}
 & & & B & \\
 & & 25 & 50 & 100 \\
\hline
 & 25 & 25 & -25 & -25 \\
A & 50 & 50 & 50 & 100 \\
 & 100 & -100 & 50 & 100 \\
\hline
\end{array}
$$

→ Compare first row with first column

→ 25+25 = 50

hence A wins B coins

= +25

then second row with second column

→ 25+50 = 75

number is odd B wins

A wins

= -25

then continue same for all other row and column

Pay off matrix for player A

(If done with reference to player B)

$$
\begin{array}{cc|ccc|}
 & & & B & \\
 & & 25 & 50 & 100 \\
\hline
 & 25 & -25 & 25 & 25 \\
A & 50 & 50 & -50 & -100 \\
 & 100 & 100 & -50 & -100 \\
\hline
\end{array}
$$

Player B will always use mini-max principle (Column)

First find the maximum value in column

## Example problems to determine saddle point

1) Solve the following game whose pay off matrix is given

$$\begin{array}{cc} & B \\ & \begin{array}{ccc} I & II & III \end{array} \\ A \begin{array}{c} I \\ II \\ III \\ IV \end{array} & \begin{bmatrix} 2 & -1 & 8 \\ -4 & -3 & 4 \\ -8 & -4 & 0 \\ 1 & -6 & -2 \end{bmatrix} \end{array}$$

→ Mini max

$$\begin{array}{c|ccc|c} & & B & & \text{Row minima} \\ A & I & II & III & \\ \hline I & 2 & -1 & 8 & -1 \\ II & -4 & -3 & 4 & -4 \\ III & -8 & -4 & 0 & -8 \\ IV & 1 & -6 & -2 & -6 \\ \hline \text{Column maxima} & 2 & -1 & 8 & \end{array}$$

Max min

→ Min max = Max-mini = -1
Hence saddle point exists

- Best strategy for player A - I

Best strategy for B - II

Value of game = -1

## Instructions-

Player A will always use max mini principle (Row)

→ First find the minimum value in row

$$\text{Ex} \rightarrow \begin{array}{c|c|c|c|c} I & 2 & -1 & 8 & -1 \leftarrow \text{Minimum value} \\ II & -4 & -3 & 4 & -4 \end{array}$$

Then compare the obtained value and select the one which has maximum (greater) value

In our case it is -1

Player B will always use mini-max principle (Column)

→ First find the maximum value in column

Ex →

| | I | II |
|---|---|---|
| I | 2 | -1 |
| II | -4 | -3 |
| | 2 | -1 |

Then compare the obtained value and select the one which has ~~maximum gre~~ minimum lesser value

Tip - Write down Row minima, Column maxima to not get confused

Example Sums on games without saddle point

1) Solve the following game whose pay off matrix is

$$\text{Player A} \quad \begin{array}{c} B \\ \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix} \end{array}$$

→

| A \ B | | | Row minima |
|---|---|---|---|
| | 3 | -2 | -2 |
| | 2 | 5 | 2 |
| Column maxima | 3 | 5 | |

$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ ← write this to not get confused

Mini max = 3 Max min = 2

There is no saddle point as Max mini is not equal to Min max

$a_{11} = 3 \qquad a_{12} = -2 \qquad a_{21} = 2 \qquad a_{22} = 5$

A

We have

$$\boxed{x_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}}$$ (Memorize)

$$= \frac{5 - (+2)}{(3+5) - (-2+2)} = \frac{3}{8}$$

$$\boxed{x_2 = 1 - x_1}$$ (Memorize)

$$= 1 - \frac{3}{8} = \frac{5}{8}$$

$$\boxed{y_1 = \frac{a_{22} - a_{12}}{(a_{22} + a_{11}) - (a_{21} + a_{12})}}$$ (Memorize)

$$= \frac{5 - (-2)}{(3+5) - (-2+2)} = \frac{7}{8}$$

$$\boxed{y_2 = 1 - y_1} = 1 - \frac{7}{8} = \frac{1}{8}$$

$$V = \frac{a_{11}a_{22} - a_{21}a_{12}}{(a_{22} + a_{11}) - (a_{21} + a_{12})} = \frac{3 \times 5 - [2 \times (-2)]}{3+5-(-2+2)}$$

$$= \frac{15+4}{8} = \frac{19}{8}$$

Hence the solution of game is

| Player | Probability of wins strategies I | II |
|---|---|---|
| A | 3/8 | 5/8 |
| B | 7/8 | 1/8 |

Value of the game: $\frac{19}{8}$

Dominance rule -

→ Row dominance - In row dominance compare Row 1 with Row 2 or 3 [or Row 2 with 3]

→ If all the values of Row 1 is greater than or equal to row 2 then Row 1 is superior hence eliminate Row 2 inferior

$R_1$ | 4 | 6 | -5 | → eliminated

$R_2$ | ~~5~~ | ~~6~~ | ~~3~~ | → $R_2$ dominates $R_1$

→ If any one value of Row 1 is lesser than any one value of row 2 then try combination of other row

Ex

$R_1$ 7 6 -5 → Since here $R_1$ has one value greater than $R_2$ (7 > 5)

$R_2$ 5 7 -3

$R_3$ ~~4 6 -5~~

we see $R_2$ and $R_3$

and since $R_2$ is greater than $R_3$

eliminate $R_3$

a

→ If all ~~the~~ Rows have one value greater than others then try column dominance first and then try row dominance

(ii) Column dominance

→ Same as row dominance but applies in columns and eliminates superior values

Ex

| | I | II | III |
|---|---|---|---|
| I | 3 | 5 | 7 |
| II | 0 | -4 | -2 |

Comparing column II and III we eliminate column III

(iii) ~~Mixed~~ Modified row dominance

→ Same principle but two rows are averaged with ~~third~~. to compare with a row

Example -

| | | |
|---|---|---|
| I | 2 | 4 |
| II | 5 | 0 |
| III | 0 | 8 |

Averaging 2nd and 3rd row = $\left(\frac{5+0}{2}\right)\left(\frac{0+8}{2}\right) = (2.5, 4)$

Comparing with first row, average is greater or equal to 1st here

| | | |
|---|---|---|
| → II | 5 | 0 |
| II | 0 | 8 |

Modified column dominance

Same as modified row dominance but eliminate superior (Row eliminates inferior)

**Problems on Dominance principle**

Using dominance principle solve the following game

~~I I II III IV~~

~~II 20 15 12 35~~

~~III~~

~~IV~~

| | I | II | III | IV |
|---|---|---|---|---|
| I | 20 | 15 | 12 | 35 |
| II | 25 | 14 | 8 | 10 |
| III | 40 | 2 | 19 | 5 |
| IV | 5 | 4 | 11 | 0 |

→

| | I | II | III | IV | Row minima |
|---|---|---|---|---|---|
| I | 20 | 15 | 12 | 35 | 12 |
| II | 25 | 14 | 8 | 10 | 8 |
| III | 40 | 2 | 19 | 5 | 2 |
| IV | 5 | 4 | 11 | 0 | 0 |
| Column maxima | 40 | 15 | 19 | 35 | |

Since max mini ≠ mini max

↳ Convert into 2×2 matrix to apply $x, y$, formula.

Using row dominance

| | I | II | III | IV |
|---|---|---|---|---|
| I | 20 | 15 | 12 | 35 |
| II | 25 | 14 | 8 | 10 |
| III | 40 | 2 | 19 | 5 |
| IV | ~~5~~ | ~~4~~ | ~~11~~ | ~~0~~ |

Using Column dominance → (we use this because row dominance cant be applied Ex- one value of row is greater than another)

| | I | II | III | IV |
|---|---|---|---|---|
| I | 20 | 15 | 12 | 35 |
| II | 25 | 14 | 8 | 10 |
| III | 40 | 2 | 19 | 5 |

Using row dominance

| | II | III | IV |
|---|---|---|---|
| I | 15 | 12 | 35 |
| II | ~~14~~ | ~~8~~ | ~~10~~ |
| III | 2 | 19 | 5 |

Using colum dominance

| | II | III | IV |
|---|---|---|---|
| I | 15 | 12 | 35 |
| III | 2 | 19 | 5 |

Hence we have

| | II | III |
|---|---|---|
| I | 15 | 12 |
| III | 2 | 19 |

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$a_{11} = 15 \quad a_{12} = 12, \quad a_{21} = 2, \quad a_{22} = 19$

$$x_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}$$

$$= \frac{19 - \cancel{1}2}{(15+19) - (2+12)} = \frac{17}{34-14} = \frac{17}{20}$$

$$x_2 = 1 - \frac{17}{20} = \frac{3}{20}$$

$$y_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})} = \frac{19 - 12}{(15+19) - (2+12)} = \frac{7}{34-14} = \frac{7}{20}$$

$$y_2 = 1 - \frac{7}{20} = \frac{13}{20}$$

$$v = \frac{a_{11}a_{22} - a_{21}a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})} = \frac{15 \times 19 - 2 \times 12}{(15+19) - (2+12)} = \frac{261}{20}$$

Solution of game is

| Player | Probability of using strategy I | III |
|---|---|---|
| A | 17/20 | 3/20 |
| B | 0 | 13/20 |

Value of game = 261/20

Use the dominance principle solve the following game

| | | |
|---|---|---|
| 1 | -3 | -2 |
| 0 | -4 | 2 |
| -5 | 2 | 3 |

| | I | II | III | IV |
|---|---|---|---|---|
| I | 3 | 2 | 4 | 0 |
| II | 3 | 4 | 2 | 4 |
| III | 4 | 2 | 4 | 0 |
| IV | 0 | 4 | 0 | 8 |

→

| | I | II | III | IV | Row minima |
|---|---|---|---|---|---|
| I | 3 | 2 | 4 | 0 | 0 |
| II | 3 | 4 | 2 | 4 | 2 |
| III | 4 | 2 | 4 | 0 | 0 |
| IV | 0 | 4 | 0 | 8 | 0 |
| Column maxima | 4 | 4 | 4 | 8 | |

As max mini ≠ mini-max

Using row dominance

| | I | II | III | IV |
|---|---|---|---|---|
| I | ~~3~~ | ~~2~~ | ~~4~~ | ~~0~~ |
| II | 3 | 4 | 2 | 4 |
| III | 4 | 2 | 4 | 0 |
| IV | 0 | 4 | 0 | 8 |

Using column dominance

$$\begin{array}{c|cccc} & \text{I} & \text{II} & \text{III} & \text{IV} \\ \text{II} & 8 & 4 & 2 & 4 \\ \text{III} & 4 & 2 & 4 & 0 \\ \text{IV} & 9 & 4 & 0 & 8 \end{array}$$

Using modified row dominance

$$\begin{array}{c|ccc} & \text{II} & \text{III} & \text{IV} \\ \text{II} & 4 & 2 & 4 \\ \text{III} & 2 & 4 & 0 \\ \text{IV} & 4 & 0 & 8 \end{array}$$

Averaging III and IV

$\left(\frac{2+4}{2}\right) = \left(\frac{4+0}{2}\right) \left(\frac{0+8}{2}\right)$

$\downarrow \quad \downarrow \quad \downarrow$

$4 \quad 2 \quad 4$

Eliminating (II)

Using modified column dominance

$$\begin{array}{c|ccc} & \text{II} & \text{III} & \text{IV} \\ \text{III} & 2 & 4 & 0 \\ \text{IV} & 4 & 0 & 8 \end{array}$$

Averaging III and IV

$\left(\frac{4+0}{2}\right) = 2 \left(\frac{0+8}{2}\right) = 4$

$$\therefore \begin{array}{c|cc} & \text{III} & \text{IV} \\ \text{III} & 4 & 0 \\ \text{IV} & 0 & 8 \end{array} \qquad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$a_{11} = 4 \quad a_{12} = 0 \quad a_{21} = 0 \quad a_{22} = 8$

$$x_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{21} - a_{12})} = \frac{8 - 0}{(4+8) - (0-0)} = \frac{8}{12} = \frac{2}{3}$$

$$x_2 = 1 - \frac{2}{3} = \frac{1}{3}$$

$$y_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}$$

$$= \frac{8}{(4+8) - (0+0)} = \frac{8}{12} = \frac{2}{3}$$

$$y_2 = 1 - \frac{2}{3} = \frac{1}{3}$$

$$V = \frac{a_{11} \times a_{22} - a_{21} \times a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})} = \frac{4 \times 8 - 0}{(4+8) - 0} = \frac{32}{12} = \frac{8}{3}$$

Solution of game is

| Player | Probability of using strategies III | IV |
|---|---|---|
| A | 2/3 | 1/3 |
| B | 2/3 | 1/3 |

Value of game $= v = \frac{8}{3}$

# Graphical Method

2 x M →

$$\begin{array}{c|cccc} & C_1 & C_2 & C_3 & C_4 \\ R_1 & 1 & 2 & 3 & 4 \\ R_2 & 9 & 7 & 0 & 5 \end{array}$$

In this row is fixed ~~colum~~ to 2 but column varies

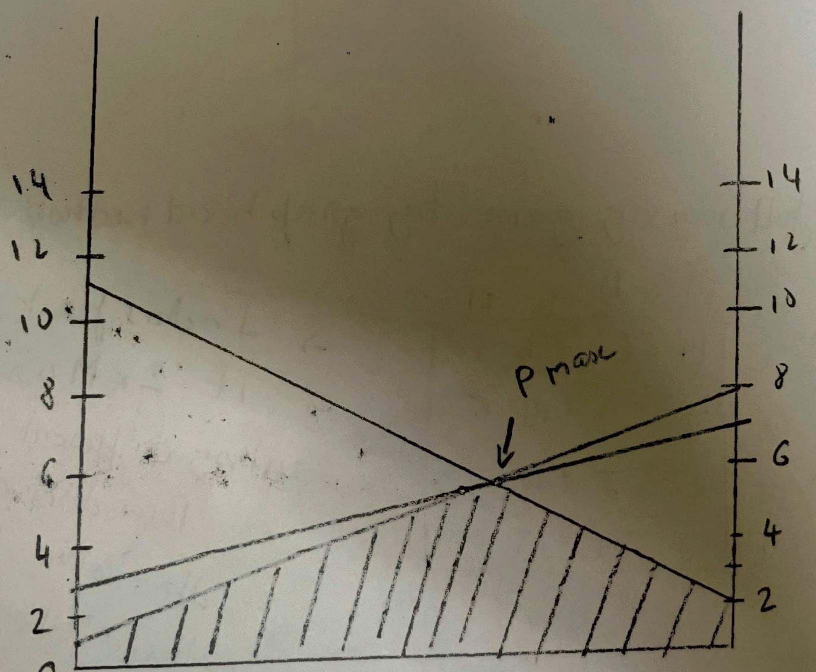
N x 2 →

$$\begin{array}{c|cc} & C_1 & C_2 \\ R_1 & 1 & 2 \\ R_2 & 3 & 4 \\ R_3 & 5 & 6 \end{array}$$

In this column is fixed to 2 but row varies
Also N x 2 we consider upper boundary
and 2 x M lower boundary

1) Solve the following 2x3 game by ~~lower boundary~~ graphical method

1 3 11
8 5 2



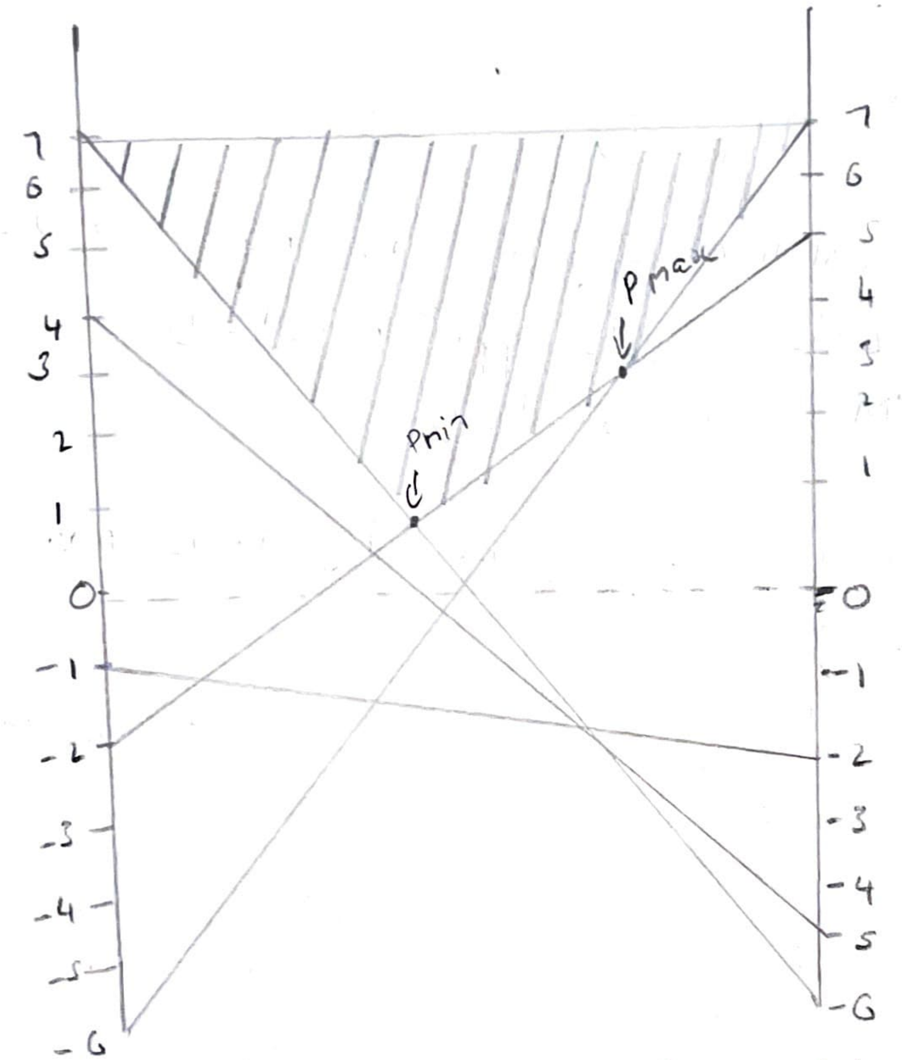

→ $\begin{bmatrix} 3 & 5 \\ 11 & 2 \end{bmatrix}$

Then use dominance rule to find game value

Solve the game by graphical method

$$\begin{bmatrix} -6 & 7 \\ 4 & -5 \\ -1 & -2 \\ -2 & 5 \\ 7 & -6 \end{bmatrix}$$

→ Bring it to N×2 matrix using dominance method




$$\begin{bmatrix} 7 & -5 \\ -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 5 \\ 7 & -6 \end{bmatrix}$$

→ Then use dominance rule to find out game value

3) Solve the following game by graphical method

| | | B | | | |
|---|---|---|---|---|---|
| | | I | II | III | IV |
| A | I | 19 | 6 | 7 | 5 |
| | II | 7 | 3 | 14 | 6 |
| | III | 12 | 8 | 18 | 4 |
| | IV | 8 | 7 | 13 | -1 |

→ In this first reduce it 2×M or N×2 using different techniques ex- Row dominance then solve by graphical method then find game value