Worked examples

1. ***A and B are playing a game with 2 coins according to the following rules.***

a. ***When both are heads (H) it is a benefit or gain of Re. 1 to player A***

b. ***When there is one head and tail (T) then it is a loss of Re. 1 to player A, and***

c. ***When there are two tails, there is no loss or gain to any player. Formulate this as pay off matrix.***

Solution:

Let H, T indicates turning of the coin as head or tail and are nothing but strategies for the players.

Rule of the play:

i) when there are two heads, Re.1 is gain to the player A

ii) when there is one head and tail it is Re.1 loss to the player A

iii) when there are two tails, no loss, no profit to any player.

The pay off matrix for the player A by following the rule for the play given we get, the formulation as below.

		B: H	B: T
A	H	1	– 1
	T	– 1	0

Pay off matrix

2. ***A and B are playing a game with 3 coins 25 p, 50 p, 100p according to the following rules.***

i) ***If the sum is even, A wins B's coin***

ii) ***If the sum is odd, B will win A's coin. Formulate this as pay off matrix.***

Solution:

Rule of the play:

i) when the sum is even A wins B's coin

ii) when the sum is odd, B will win A's coin

The pay off matrix for the player A by following the rule for the play given we get the pay off matrix as below

		B: 25	B: 50	B: 100
A	25	25	– 25	– 25
	50	– 50	50	100
	100	– 100	50	100

In a game played by fingers two players A and B are simultaneously showing 2 or 3 fingers. When the sum of the fingers is odd A gets the points equal to the sum. When the sum of fingers is even A loses the points equal to the sum. Write the pay off matrix.

...tion:

...showing 2 or 3 fingers be the strategies for the players.

...le of the play:

If the sum of the fingers is odd the player A gains the sum

If the sum is even he loses the sum to player B.

The pay off matrix for the player A considering the rule of the play is,

		B: 2	B: 3
A	2	−4	5
	3	5	−6

4. ***Solve the following two person zero sum game with the following 3 × 2 pay off matrix for player A.***

$$\begin{array}{cc} & \textbf{Player } B \\ & \begin{array}{cc} B_1 & B_2 \end{array} \\ \textbf{Player } A \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 9 & 2 \\ 8 & 6 \\ 6 & 4 \end{bmatrix} \end{array}$$

Solution:

Step 1: Write the row minima against each strategy (A_1, A_2, A_3) and identify the highest among these (That is 6).

$$\begin{array}{cccc} & B_1 & B_2 & \text{Row minima} \\ A_1 & 9 & 2 & 2 \\ A_2 & 8 & 6 & \boxed{6}\ \text{Maxi min} \\ A_3 & 6 & 4 & 4 \\ \text{Column maxima} & 9 & \boxed{6}\ \text{Mini max} & \end{array}$$

Step2: Write the column maxima against each strategy (B_1, B_2) and identify the minimum element between these two (that is 6).

Thus, Mini max = Maxi min

Hence, the game is having saddle point. Identify the optimal strategies for the both the players corresponding to these values by drawing horizontal and vertical lines. The intersection of these two lines gives the value of game.

i. The optimal strategy for player A is A_2.

ii. The optimal strategy for player B is B_2 and

iii. The value of the game = 6

Observation

Though player A is having three strategies and player B 2 strategies, they are using a single strategy. Hence the nature of the game is deterministic / strictly determinable in nature (pure strategy game).

5. *Solve the game whose payoff matrix is given by,*

$$\begin{array}{cc} & \textit{Player } B \\ & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \textit{Player } A \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{bmatrix} \end{array}$$

Solution:

Let us find Min max and Maxmin

Player B

		B_1	B_2	B_3	Row minima	
Player A	A_1	1	3	1	[1]	Maxi min
	A_2	0	–4	–3	–4	
	A_3	1	5	–1	–1	
Column maxima		[1]	5	[1]	Mini max	

Mini max = 1 Maxi min = 1

Hence, the game is having saddle point

i. The optimal strategy for player A is A_1

ii. The optimal strategy for player 'B' is B_1 or B_3

iii. The value of game = + 1

	B_1	B_2	B_3	
A_1	1	3	1	[1]
A_2	0	– 4	– 3	– 4
A_3	1	5	– 1	– 1
	[1]	5	1	

or

	B_1	B_2	B_3	
A_1	1	3	1	1
A_2	0	– 4	– 3	– 4
A_3	1	5	– 1	– 1
	1	5	[1]	

Note: This game is having alternative optimal solution as the optimal strategy for player B may be B_1 or B_3.

6. ***Solve the following game whose pay off matrix is given below***

B

		I	***II***	***III***
A	***I***	***–3***	***–2***	***6***
	II	***2***	***0***	***2***
	III	***5***	***–2***	***–4***

Solution:

Let us find Min max and Maxmin

	I	II	III	Row minima	
I	–3	–2	6	–3	Maxi min
II	2	[0]	2	[0]	
III	5	–2	–4	–4	
Column maxima	5	[0]	6		
		Mini max			

We have Maxi min = Mini max, therefore saddle point exists. The game is deterministic in nature(saddle point exists).

Best strategy for A = II

Best strategy for B = II

Value = 0

The above game is referred as a fair game as the value of it is zero.

7. ***Solve the following game whose pay off matrix is given in the following matrix.***

		B		
		I	*II*	*III*
A	*I*	*2*	*− 1*	*8*
	II	*− 4*	*− 3*	*4*
	III	*− 8*	*− 4*	*0*
	IV	*1*	*− 6*	*− 2*

Solution:

	I	II	III	Row minima
I	2	−1	8	−1 Maxi min
II	−4	−3	4	−4
III	−8	−4	0	−8
IV	1	−6	−2	−6
Column maxima	2	−1 Mini max	8	

Min Max=Maxmin= -1, Hence there exists saddle point.

Best strategy for A = I

Best strategy for B = II

Value of game = − 1

Whenever, Maxi min = Mini max, there exists saddle point or equilibrium point and the game is deterministic or pure strategy game.

Note: Player A will try to maximize his minimum gains and player B will try to minimize his maximum losses. Hence Maxi min is applicable to player A and Mini max is applicable to player B

8. ***Consider the game G with the following pay off. Determine the value of game ignoring the value of λ.***

	B_1	B_2
A_1	2	6
A_2	− 2	λ

Solution:

Ignoring the value of λ,

		Row minima
2	6	2 Maxi min
– 2	λ	– 2
Column maxima: 2 (Mini max)	6	

As Maxi min = Mini max, saddle point exists.

Best strategy for row player = A_1 Best strategy for column player = B_1

Value of the game = 2

9. ***For what value of λ, the game with the following pay off matrix is strictly determinable (pure strategy).***

A \ *B*	*I*	*II*	*III*
I	λ	6	2
II	– 1	λ	– 7
III	– 2	4	λ

Solution:

Ignoring the value of λ whatever it may be, we get

			Row minima
λ	6	2	2 *Maxi min*
–1	λ	–7	–7
–2	4	λ	–2
Column maxima –1 (*Mini max*)	6	2	

Maxi min = 2, Mini max = – 1.

For strictly determinable the value of λ is $-1 \leq \lambda \leq 2$.

That is in the range between Maxi min and Mini max values)

10. ***Determine the range of values of p and q that will make the pay off matrix (a_{ij}) given below, a deterministic game in nature.***

$$\text{Player } A \overset{\text{Player } B}{\begin{bmatrix} 2 & 4 & 7 \\ 10 & 7 & q \\ 4 & p & 8 \end{bmatrix}}$$

...ution:

Let us first consider the game to determine Maxi. min and Mini. max ignoring the values p and q

$$\begin{array}{c c c c c} & B_1 & B_2 & B_3 & \text{Row minima} \\ A_1 & 2 & 4 & 7 & 2 \\ A_2 & 10 & 7 & q & \boxed{7} \text{ Maxi min} \\ A_3 & 4 & p & 8 & 4 \\ \text{Column maxima} & 10 & \boxed{7} & 8 & \\ & & \text{Mini max} & & \end{array}$$

Thus the Maxi min = Mini max = 7

Thus there exists a saddle point at position (2, 2).

This imposes / requires the condition on 'p' as $p \leq 7$ and q as $q \geq 7$

Hence, $p \leq 7$, $q \geq 7$ is the range of 'p' and 'q'.

$(a_{11}+a_{22})-(a_{21}+a_{12})$

11. *Solve the following game whose pay off matrix is*

$$\text{Player } A \overset{\text{Player } B}{\begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix}}$$

Solution:

Given game is

$$\begin{array}{cc} & \begin{array}{ccc} & B & \\ I & II & \text{Row minima} \end{array} \\ A\begin{array}{c} I \\ II \end{array} & \left[\begin{array}{cc} 3 & -2 \\ 2 & 5 \end{array}\right]\begin{array}{c} -2 \\ 2 \end{array} \\ \text{Column maxima} & \begin{array}{cc} 3 & 5 \end{array} \end{array}$$

Max.min = 2, Min max = 3

There is no saddle point as Max min is not equal to Min max

Hence, the nature of game is probabilistic and the value of the game lies in between 2 and 3 (in between Max min and Min max).

Let x_1, x_2 the probabilities of using I, II strategies by player 'A' and y_1, y_2 be the probabilities of using I, II strategies by player 'B' and 'v' be the value of game.

Comparing the elements of the game with $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

We have,

$$x_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}, \quad x_2 = 1 - x_1$$

$$y_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}, \quad y_2 = 1 - y_1$$

and $$v = \frac{a_{11}a_{22} - a_{21}a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}$$

Substituting the values of a_{11}, a_{12}, a_{21}, a_{22} in the above equations we get,

$$x_1 = \frac{5-(+2)}{(3+5)-(-2+2)} = \frac{3}{8}$$

$$x_2 = 1 - x_1 = 1 - \frac{3}{8} = \frac{5}{8}$$

$$y_1 = \frac{5-(-2)}{(3+5)-(-2+2)} = \frac{7}{8}$$

$$y_2 = 1 - y_1 = 1 - \frac{7}{8} = \frac{1}{8}$$

and, $v = \frac{(3\times5)-[2\times(-2)]}{(3+5)-(-2+2)} = \frac{15+4}{8} = \frac{19}{8}$ Hence, the solution of the game is,

Player	Probabilities of using strategies	
Player	I	II
A	$\frac{3}{8}$	$\frac{5}{8}$
B	$\frac{7}{8}$	$\frac{1}{8}$

Value of game (for player 'A') $= \frac{19}{8}$

12. *In a game of matching coins, player 'A' wins Rs. 8, if both coins show heads and Rs. 1 if both are tails. Player B wins Rs. 3 when coins do not match. Given the choice of being Player A or Player B, which would you choose and what would be your strategy?*

Solution:

Let showing head, tail of the coin be the strategies as H, T

If both are heads 'A' will gain 8 points, if there is one head and one tail player 'A' will get 1 point and if both are the tail 'A' looses 3 points.

The formulation of the game is,

$$\begin{array}{c} \\ \\ A \end{array}\begin{array}{cc} & B \\ & \begin{array}{cc} H & T \end{array} \\ \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} 8 & -3 \\ -3 & 1 \end{bmatrix} \end{array}$$

Comparing the formulated game with

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

We have,

Let x_1, x_2 be the probabilities of showing head, tail by player 'A', Let y_1, y_2 be the probabilities of showing head, tail by player 'B' respectively

$$x_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}, \quad x_2 = 1 - x_1$$

$$y_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}, \quad y_2 = 1 - y_1$$

and the value of game,

$$v = \frac{a_{11}a_{22} - a_{21}a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}$$

Substituting the values of a_{11}, a_{12}, a_{21} and a_{22} in the above equations we get,

$$x_1 = \frac{(1)-(-3)}{(8+1)-(-3-3)} = \frac{4}{9-(-6)} = \frac{4}{15}$$

$$x_2 = 1 - x_1 = 1 - \frac{4}{15} = \frac{11}{15}$$

$$y_1 = \frac{(1)-(-3)}{(8+1)-(-3-3)} = \frac{1+3}{9-(-6)} = \frac{4}{15}$$

$$y_2 = 1 - \frac{4}{15} = \frac{11}{15}$$

$$v = \frac{(8)(1)-(-3\times-3)}{(8+1)-(-3+-3)} = \frac{8-9}{9+6} = \frac{-1}{15}$$

Hence, the solution of the game is

Player	Probabilities of using	
	Head	Tail
A	$\frac{4}{15}$	$\frac{11}{15}$
B	$\frac{4}{15}$	$\frac{11}{15}$

and value of the game is $\frac{-1}{15}$.

Choice is to be player 'B' as 'A' is the looser as per the value of game. *Strategies*: The probabilities of using head and tail of the coin are 4/15, 11/15.

13. ***Solve the following game by using the dominance concept***

		Player B		
		B_1	B_2	B_3
Player A	A_1	4	5	8
	A_2	6	4	6
	A_3	4	2	4

Solution:

Let us find the row minima and column maxima

		Player B			
		B_1	B_2	B_3	Row minima
Player A	A_1	4	5	8	4
	A_2	6	4	6	4 Maxi min
	A_3	4	2	4	2
column maxima		6	5	8	Mini max

Since Maxi min ≠ Mini max, we can't use the method of pure strategy (saddle point concept) to solve the game.

It can be seen that row A_2 dominates row A_3 as every element of row $A_2 \geq$ row A_3, thus row A_3 can be deleted. The resulting matrix is

	B_1	B_2	B_3
A_1	4	5	8
A_2	6	4	6

Column B_1 dominates column B_3 as the element values of $B_1 \le$ element of B_3

Hence B_3 can be deleted. The reduced matrix is,

	B_1	B_2
A_1	4	5
A_2	6	4

Let the probabilities of using the strategies A_1, A_2, A_3, be x_1, x_2, x_3 respectively and the probabilities of using B_1, B_2, B_3 strategies by player 'B' be y_1, y_2, y_3 respectively such that $x_1 + x_2 + x_3 = 1$

$y_1 + y_2 + y_3 = 1$

In the reduced matrix $a_{11} = 4,\ a_{12} = 5,\ a_{21} = 6,\ a_{22} = 4$

Then using the formula,

$$x_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{21} + a_{12})} \qquad x_2 = 1 - x_1$$

$$y_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})} \qquad y_2 = 1 - y_1$$

$$v = \frac{a_{11}\,a_{22} - a_{21}\,a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})},$$ Substituting the values of $a_{11}, a_{12}, a_{21}, a_{22}$ in the above equations,

we get,

$$x_1 = \frac{2}{3},\ x_2 = \frac{1}{3},\ x_3 = 0,\ y_1 = \frac{1}{3},\ y_2 = \frac{2}{3},\ y_3 = 0$$

and the value of game $v = \dfrac{14}{3}$

i. Probabilities of using I, II, III strategies by the player A

$$x_1 = \frac{2}{3},\ x_2 = \frac{1}{3},\ x_3 = 0$$

ii. Probabilities of using I, II, III strategies by the player B

$$y_1 = \frac{1}{3},\ y_2 = \frac{2}{3},\ y_3 = 0$$

The value of game $v = \dfrac{14}{3}$

14. *Determine the nature of the following game, whose pay off matrix is,*

		B I	B II	B III
A	I	−1	2	1
	II	1	−2	2
	III	3	4	−3

Solution

	I	II	III	Row minima
I	−1	2	1	−1 (Maxi min)
II	1	−2	2	−2
III	3	4	−3	−3
Column maxima	3	4	2 (Mini max)	

Since the Mini max ≠ Maxi min, there is no saddle point that is, it is probabilistic game or mixed strategy game.

15. ***Solve the following 2 person zero sum game based on the concept of dominance.***

$$\begin{array}{c} \\ I \\ II \\ III \end{array}\begin{array}{c} \begin{array}{ccc} I & II & III \end{array} \\ \begin{bmatrix} -4 & 6 & 3 \\ -3 & -3 & 4 \\ 2 & -3 & 4 \end{bmatrix} \end{array}$$

Solution

Let x_1, x_2 and x_3 be the probabilities of I, II and III strategies for player A y_1, y_2 and y_3 be the probabilities for player 'B'.

It can be observed that all the elements of $R_3 \geq$ corresponding elements of R_2. Hence, delete R_2.

The reduced matrix will be,

	I	II	III
I	−4	6	3
III	2	−3	4

Comparing column wise all the elements of 3rd column ≥ the corresponding elements of 1st column.

So, the player 'B' will never use 3rd strategy

Thus, the reduced matrix is,

	I	II
I	− 4	6
III	2	− 3

Comparing the reduced matrix with $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and on solving we get,

(As, the game is 2 × 2 game without saddle point)

$$x_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{12})} = \frac{1}{3},\ x_2 = \frac{2}{3}, x_3 = 0$$

(Sum of the probabilities = 1)

$$y_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{12})} = \frac{3}{5},\ y_2 = \frac{2}{5}, y_3 = 0$$

and

$$\text{Value of game } v = \frac{a_{11}a_{22} - a_{21}a_{22}}{(a_{11} + a_{22}) - (a_{21} + a_{12})} = 0$$

∴ It is a fair game

Thus, the optimal strategies are,

For player A (I, II, III) with probabilities $\left(\frac{1}{3},\ 0,\ \frac{2}{3}\right)$

For player B (I, II, III) with probabilities $\left(\frac{3}{5},\ \frac{2}{5},\ 0\right)$

Note: *Dominance property is used when the saddle point does not exists. Prior to applying dominance property check for the existence of the saddle point, find the Max. Min and Min. Max to check whether the problem is having saddle point or not.*

16. ***Using the dominance concept, obtain the optimal strategies for both the players and determine the value of game. The pay off matrix for player A is given.***

		B				
		I	*II*	*III*	*IV*	*V*
	I	2	4	3	8	4
	II	5	6	3	7	8
A	*III*	6	7	9	8	7
	IV	4	2	8	4	3

Solution:

This problem is having saddle point (min max = max min = 6). But as it is given to use the dominance property, let us solve it by dominance rule only. By inspection of rows. It is clear that row III dominates row IV as all the elements of row III are ≥ row IV. Hence, row IV can be deleted. The resulting matrix is

		B I	II	III	IV	V
	I	2	4	3	8	4
A	II	5	6	3	7	8
	III	6	7	9	8	7

We can see that column I dominates column IV as all the elements of column I are ≤ column IV. Hence, column IV can be deleted. The resulting matrix is,

		B I	II	III	V
	I	2	4	3	4
A	II	5	6	3	8
	III	6	7	9	7

It can be seen that row III dominates row I. Hence, delete row I.

The resulting matrix is

		B I	II	III	V
A	II	5	6	3	8
	III	6	7	9	7

Column I dominates column V

Hence, deleting column V we get

		B I	II	III
A	II	5	6	3
	III	6	7	9

Column I dominates column II

Hence delete column II. The resulting matrix is

B

A	I	III
I	5	3
III	6	9

Row I dominates row II, deleting it we get,

B

A	I	II
III	6	9

Again column I dominates column II

Hence, delete column II. Thus,

B

A	I
III	6

is the reduced matrix (element).

Best strategy for player – A III

Best strategy for player – B I and the value of game = 6

Note:

i) *The same answer can be verified with saddle point concept*

ii *If the dominance principle is applied to the pay-off matrix having a saddle point, then we get a single element reduced matrix only.*

17. ***Solve the following game by dominance concept.***

B

A	*I*	*II*	*III*
I	1	7	2
II	6	2	7
III	5	2	6

Solution

Step 1

All the elements of row 2 ≥ row 3 and row 2 is superior to row 3. Hence delete row 3.

1	7	2
6	2	7

Step 2

As all the elements of column 1 ≤ column 3 and column 1 is superior than column 3. Hence, delete column 3.

	I	II
I	1	7
II	6	2

$$x_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{21} + a_{12})} = \frac{2-6}{(3)-(6+7)} = \frac{-4}{-10} = \frac{2}{5} \qquad x_2 = \frac{3}{5}$$

$$y_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})} = \frac{2-7}{-10} = \frac{-5}{-10} = \frac{1}{2} \qquad y_2 = \frac{1}{2}$$

(Sum of the probabilities is unity)

$$v = \frac{a_{11}\, a_{22} - a_{21}\, a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})} = \frac{2-42}{-10} = \frac{-40}{-10} = 4$$

Probabilities of using I, II and III strategies by player A (x_1, x_2, x_3) is $\left(\frac{2}{5}, \frac{3}{5}, 0\right)$

Probabilities of using I, II and III strategies by player B (y_1, y_2, y_3) is $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$

Value of the game is = 4

18. ***Use the dominance principle solve the following game.***

		B			
		I	*II*	*III*	*IV*
	1	*20*	*15*	*12*	*35*
A	*2*	*25*	*14*	*8*	*10*
	3	*40*	*2*	*19*	*5*
	4	*5*	*4*	*11*	*0*

Solution:

B

		I	II	III	IV	Row minima
	1	20	15	12	35	12
	2	25	14	8	10	8
A	3	40	2	19	5	2
	4	5	4	11	0	0
Column maxima		40	15	19	35	

There is no saddle point

Therefore it is mixed strategy problem [The value of the game between 12 and 15]

Row dominance

Therefore row one is set dominate row – 4, eliminating row – 4 we get

B

		I	II	III	IV
	1	20	15	12	35
A	2	25	14	8	10
	3	40	2	19	5

Column dominance

Column II is set to be dominating column I. Therefore eliminating column I we get,

	II	III	IV
1	15	12	35
2	14	8	10
3	2	19	5

Again row 1 is dominates row- 2, deleting row -2 we get

	II	III	IV
1	15	12	35
3	2	19	5

Column 2 is set to dominate column 4, eliminating column 4 we get,

	II	III
1	15	12
3	2	19

Comparing the reduced game with

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

we have,

Let x_1, x_2, x_3, x_4 be the probabilities for player 'A', Let y_1, y_2, y_3, y_4 be the probabilities of for the player 'B' of the strategies respectively

Substituting the values of a_{11}, a_{12}, a_{21} and a_{22} in the formulae we get,

$$\text{Strategies of } A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \frac{17}{20} & 0 & \frac{3}{20} & 0 \end{pmatrix}$$

$$\text{Strategies of } B = \begin{pmatrix} I & II & III & IV \\ 0 & \frac{7}{20} & \frac{13}{20} & 0 \end{pmatrix}$$ and the value of game

$$V = \frac{(12 \times 17) + (19 \times 3)}{17 + 3} = \frac{204 + 57}{20} = \frac{261}{20}$$

19. *Use the dominance principle, solve the following game.*

		B		
		I	II	III
A	1	1	−3	−2
	2	0	−4	2
	3	−5	2	3

Solution:

		B			
		I	II	III	Row minima
A	1	1	−3	−2	−3
	2	0	−4	2	−4
	3	−5	2	3	−5
Column maxima		1	2	3	

As there is no saddle point, as Max min is not equal to Min max. It is a mixed strategy game

Column Dominance

Column II dominates column III, eliminating column II we get,

		B I	B II
A	1	1	– 3
	2	0	– 4
	3	– 5	2

Row dominance

Row 1 dominates row 2, eliminating row 2 we get,

	I	II
1	1	– 3
3	– 5	2

Comparing the formulated game with

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

we have,

Let x_1, x_2, x_3 be the probabilities for player 'A', Let y_1, y_2, y_3 be the probabilities for player 'B' respectively

Substituting the values of a_{11}, a_{12}, a_{21} and a_{22} in the formulae we get,

$$\text{Strategies of } A = \begin{pmatrix} 1 & 2 & 3 \\ \frac{7}{11} & 0 & \frac{4}{11} \end{pmatrix}$$

$$\text{Strategies of } B = \begin{pmatrix} I & II & III \\ \frac{5}{11} & \frac{6}{11} & 0 \end{pmatrix} = \frac{1\times 7+(-5)4}{7+4} = \frac{-13}{11}$$

20. *Solve the following game by using the concept of dominance*

		B			
		I	*II*	*III*	*IV*
A	*I*	*3*	*2*	*4*	*0*
	II	*3*	*4*	*2*	*4*
	III	*4*	*2*	*4*	*0*
	IV	*0*	*4*	*0*	*8*

Solution:

					Row minima
	3	2	4	0	0
	3	4	2	4	2 Maxi min
	4	2	4	0	0
	0	4	0	8	0
Column maxima	4 (Mini max)	4	4	8	

As there is no saddle point let us use the dominance concept

All the elements of $R_3 \geq R_1$, R_3 is superior, delete R_1

	I	II	III	IV
II	3	4	2	4
III	4	2	4	0
IV	0	4	0	8

All the elements of $C_3 \leq C_1$, C_1 is inferior, C_3 is superior, delete C_1.

	II	III	IV
II	4	2	4
III	2	4	0
IV	4	0	8

There is no pure dominance of rows or columns. Hence, average of two rows or columns can be considered.

The average is C_3 and C_4 $\left(\frac{2+4}{2}, \frac{4+0}{2}, \frac{0+8}{2}\right) \leq C_2$

Therefore, C_2 is inferior. Hence, delete C_2.

	III	IV
II	2	4
III	4	0
IV	0	8

The average of R_3 and $R_4 \geq R_2$, R_2 is inferior, hence deleting R_2 we get,

	III	IV
III	4	0
IV	0	8

Let x_1, x_2, x_3, x_4 be the probability of using I, II, III & IV strategies by player A and y_1, y_2, y_3, y_4 be the probabilities of using I, II, III & IV by player B. As strategies, I, II of both the players are deleted we get,

$$x_3 = \frac{8-0}{12-0} = \frac{8}{12} = \frac{2}{3} \qquad x_4 = \frac{1}{3}$$

$$y_3 = \frac{8-0}{12-0} = \frac{8}{12} = \frac{2}{3} \qquad y_4 = \frac{1}{3}$$

$$v = \frac{32-0}{12} = \frac{32}{12} = \frac{8}{3}$$

	Strategies			
	I	II	III	IV
Probabilities for A	0	0	2 / 3	1 / 3
Probabilities for B	0	0	2 / 3	1 / 3
Value of game is =	8/ 3			

22. *Solve the following game by graphical method*

	B_1	B_2	B_3	B_4
A_1	2	2	3	– 1
A_2	4	3	2	6

Solution:

The game does not have a saddle point. Hence it is probabilistic in nature. Player A is having two strategies while player B has three strategies. Player A has t0 choose two best strategies of B. let x_1, x_2 be two probabilities of using the I, II strategies respectively.

$x_1 + x_2 = 1$; $\quad x_2 = 1 - x_1$ $\quad$ [sum of the probabilities is unity].

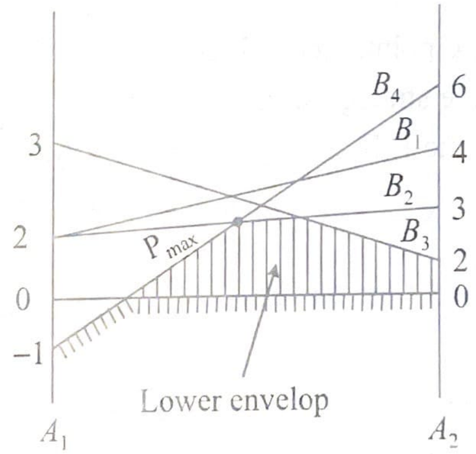
Step 1: The given game is of 2 × 4(one player is having 2 strategies), hence graphical method can be used.

Step 2: Draw two parallel lines to include the boundaries of two strategies of first player; say 'A' (Any convenient distance may be taken to draw parallel lines).

These two lines will represent two strategies available to the player "A'.

Step 3: Draw lines to represent each of player B's strategies. To represent the player B's strategy, join point 2on axis I to point 4 on axis II, point 2 on axis I to point 3 on axis II and so on.

Step 4: The game is of $2 \times M$ type, hence identify the highest point in the lower envelop P_{max} and passing through this point the corresponding strategies of the column player are II and III hence, the reduced matrix is



B_2, B_3 and B_4 have the strategies at point P_{max}. Hence delete other strategies of B. Consider B_2, B_3.

Step 5: The reduced matrix will be

B_2	B_4

A_1	2	– 1
A_2	3	6

The above game is 2 × 2 game without saddle point.

Let x_1, x_2 are the probabilities of using I, II strategies by player A.

Let y_1, y_2, y_3, y_4 are the probabilities of using I, II, III, IV strategies by player B.

$$x_1 = \frac{2-3}{4-6} = \frac{-1}{-2} = \frac{1}{2},\ x_2 = \frac{1}{2}$$

$$y_1 = \frac{2-3}{4-6} = \frac{-1}{-2} = \frac{1}{2},\ y_2 = \frac{1}{2}$$

$$v = \frac{4-9}{4-6} = \frac{-5}{-2} = \frac{5}{2}$$

$$(x_1, x_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

$$(y_1, y_2, y_3, y_4) = \left(0, \frac{1}{2}, 0, \frac{1}{2}\right)$$

23. ***Solve the following game by graphical method***

		B				
		I	*II*	*III*	*IV*	*V*
	1	*3*	*0*	*6*	*– 1*	*7*
A	*2*	*– 1*	*5*	*– 2*	*2*	*1*

Solution:

The game does not have a saddle point. Hence it is probabilistic in nature. Player A is having two strategies while player B has three strategies. Player A has to choose two best strategies of B. let x_1, x_2 be two probabilities of using the I, II strategies respectively.

$x_1 + x_2 = 1$; $x_2 = 1 - x_1$ [sum of the probabilities is unity].

Step1*:* The given game is of 2 × 5 (one player is having 2 strategies), hence graphical method can be used.

Step 2*:* Draw two parallel lines to include the boundaries of two strategies of first player; say 'A' (Any convenient distance may be taken to draw parallel lines).

These two lines will represent two strategies available to the player "A'.

Step3*:* Draw lines to represent each of player B's strategies. To represent the player B's strategy, join point 3 on axis I to point -1 on axis II, point 0 on axis I to point 5 on axis II and so on.

The feasible region is the lower boundary as shown, the highest point is shown by Max point.

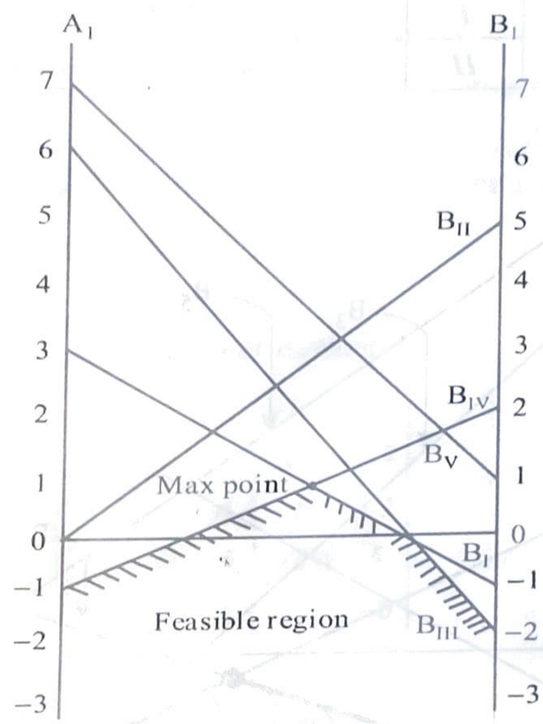
The strategies passing through this point are I and IV. Thus, the reduced game is,

	I	IV
1	3	− 1
2	− 1	2

Comparing the reduced matrix with $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and applying the following formulae

Let x_1, x_2 are the probabilities of using I, II strategies by player A.

Let y_1, y_2, y_3, y_4 are the probabilities of using I, II, III, IV strategies by player B.



$$x_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}$$

$$y_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})} \text{ and}$$

Value of game $v = \frac{a_{11}a_{22} - a_{21}a_{22}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}$

Substituting the values of a_{11}, a_{12}, a_{21} and a_{22} in the above equations we get,

Strategies of $A = \left(\frac{3}{4}, \frac{4}{7}\right)$

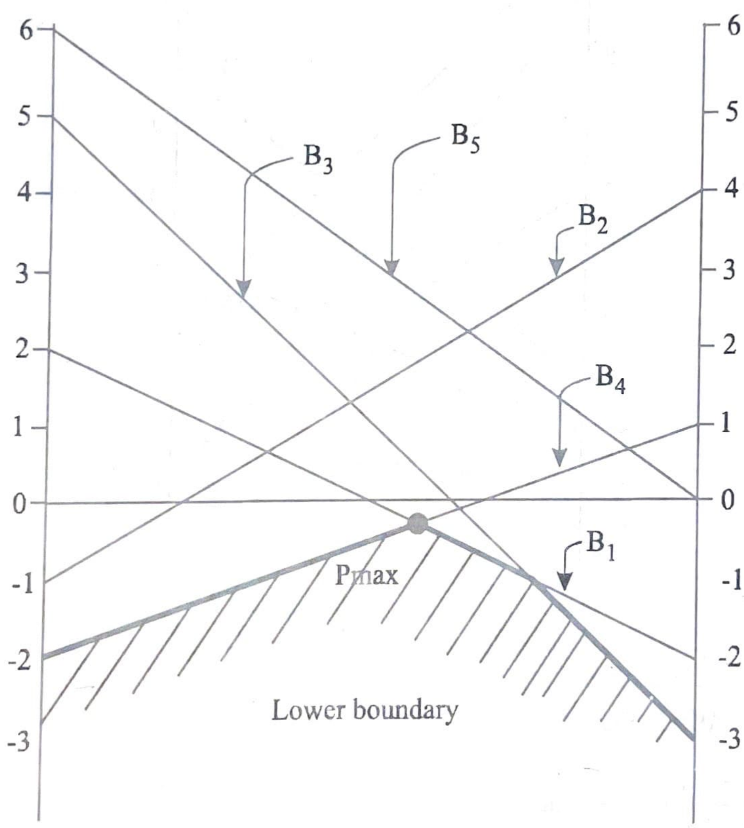
Strategies of $A = \left(\frac{3}{7},\ 0,\ 0,\ \frac{4}{7},\ 0\right)$

$$V = \frac{-1\times3+2\times4}{7+4} = \frac{5}{11}$$

24. ***Reduce the following (2 × n) game to (2 × 2) game by graphical method and hence solve.***

		B				
		I	*II*	*III*	*IV*	*V*
A	*I*	*2*	*-1*	*5*	*-2*	*6*
	II	*-2*	*4*	*-3*	*1*	*0*

Solution:



The graph represent the pay-off lines, identify the lower boundary and locate the highest point P_{max} on it (Max. min principle). Selecting the strategies passing through this point we get the reduced matrix/game (2 × 2) as,

		B	
		I	IV
A	I	2	-2
	II	-2	1

5. *Solve the following game using graphical method*

$$\begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{matrix}\begin{bmatrix} -6 & 7 \\ 4 & -5 \\ -1 & -2 \\ -2 & 5 \\ 7 & -6 \end{bmatrix}$$

Solution:

The game does not have a saddle point. Hence it is probabilistic in nature. Player B is having two strategies while player A has five strategies. Player B has to choose two best strategies of A. let y_1, y_2 be two probabilities of using the I, II strategies respectively.

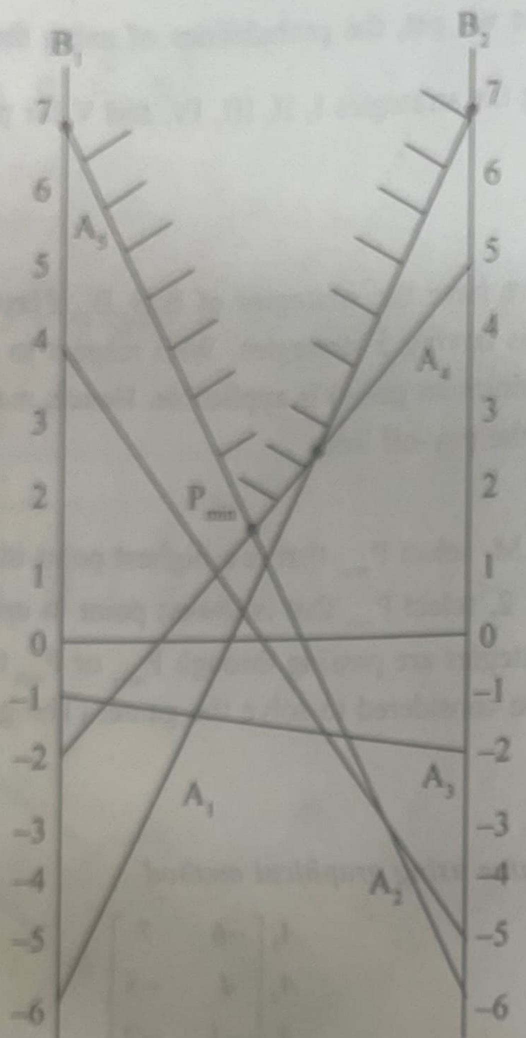
$y_1 + y_2 = 1$; $y_2 = 1 - y_1$ [sum of the probabilities is unity].

Step 1: The given game is of 5 × 2 (one player is having 2 strategies), hence graphical method can be used.

Step 2: Draw two parallel lines to include the boundaries of two strategies of column player; say 'B' (Any convenient distance may be taken to draw parallel lines).

These two lines will represent two strategies available to the player "B'.

Step 3: Draw lines to represent each of player A's strategies. To represent the player A's strategy, join point -6 on axis I to point 7 on axis II, point 4 on axis I to point -5 on axis II and so on.



P_{min} is the lowest point in the upper boundary, passing through this point we have the strategies A_4, A_5.

Step 5: The reduced matrix is

$$\begin{array}{c} \\ A_4 \\ A_5 \end{array}\begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} -2 & 5 \\ 7 & -6 \end{bmatrix} \end{array}$$

Let x_1, x_2, x_3, x_4, x_5 be the probabilities of using I, II, III, IV, V strategies by player 'A' and y_1, y_2 be the probabilities of using I, II strategies by player B.

$x_1 = x_2 = x_3 = 0$, as they are eliminated from the graph. Comparing the reduced game with

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

we have,

$$x_4 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}, \quad x_5 = 1 - x_4$$

$$y_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}, \quad y_2 = 1 - y_1$$

and $$v = \frac{a_{11}a_{22} - a_{21}a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}$$

Substituting the values of a_{11}, a_{12}, a_{21}, a_{22} in the above equations we get,

$$x_4 = \frac{-6-7}{-8-12} = \frac{-13}{-20} = \frac{13}{20}$$

$$x_5 = 1 - x_4 = 1 - \frac{13}{20} = \frac{7}{20}$$

$$y_1 = \frac{-6-5}{-8-12} = \frac{-11}{-20} = \frac{11}{20}$$

$$y_2 = 1 - \frac{11}{20} = \frac{9}{20}$$

$$v = \frac{12-35}{-8-12} = \frac{-23}{-20} = \frac{23}{20}$$

26. *Using graphical method solve the following game.*

		B	
		I	II
	1	1	2
A	2	5	6
	3	– 7	– 9
	4	– 4	– 3
	5	2	1

Solution:

The game does not have a saddle point. Hence it is probabilistic in nature. Player B is having two strategies while player A has five strategies. Player B has to choose two best strategies of A. let y_1, y_2 be two probabilities of using the I, II strategies respectively.

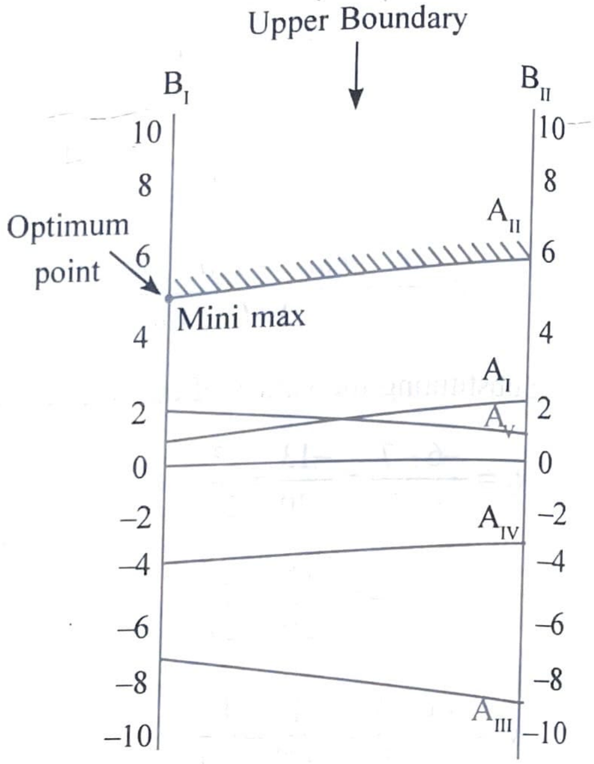
$y_1 + y_2 = 1$; $y_2 = 1 - y_1$ [sum of the probabilities is unity].

Step 1: The given game is of 5 × 2 (one player is having 2 strategies), hence graphical method can be used.

Step 2: Draw two parallel lines to include the boundaries of two strategies of column player; say 'B' (Any convenient distance may be taken to draw parallel lines).

These two lines will represent two strategies available to the player "B'.

Step 3: Draw lines to represent each of player A's strategies. To represent the player A's strategy, join point -6 on axis I to point 7 on axis II, point 4 on axis I to point -5 on axis II and so on.



From the graph, the minimum point in the upper boundary is shown by Mini max. This point is optimum point and corresponds to II strategy. Hence, the game is reduced to a single element. In other words the game is having *saddle point* and hence is deterministic in nature

27. ***Solve the following game by graphical method***

		B			
		I	*II*	*III*	*IV*
A	*I*	*19*	*6*	*7*	*5*
	II	*7*	*3*	*14*	*6*
	III	*12*	*8*	*18*	*4*
	IV	*8*	*7*	*13*	*– 1*

Solution:

The given game is of *M* × *N* matrix let us reduced it either 2 × *M* or *N* × 2 by using dominance principle then graphical method can be used to reduce it to 2 × 2 game.

All the elements of row III ≥ row IV.

Hence row III is superior, delete row IV.

	I	II	III	IV
I	19	6	7	5
II	7	3	14	6
III	12	8	18	4

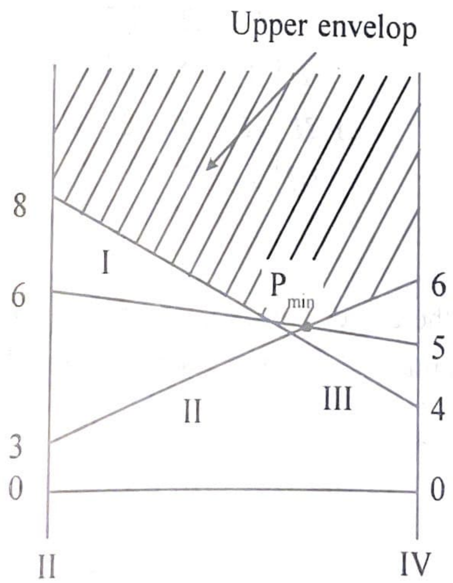
All the elements of column II ≤ column I, column II is superior, discard column I.

	II	III	IV
I	6	7	5
II	3	14	6
III	8	18	4

All the elements of column II ≤ column III, column II is superior, discard column III.

	II	IV
I	6	5
II	3	6
III	8	4

Let x_2, x_4 be he probabilities of using the strategies I, IV by player 'A' and x_1, x_2, x_3 can be probabilities of using I, II, III strategies by player 'A'.The game is reduced to 3 × 2 for which graph is plotted



From graph we get,

		B	
		II	IV
A	I	6	5
	II	3	6

The above game is 2 × 2 without saddle point.

Let x_1, x_2, x_3, x_4 are the probabilities of using I, II, III, IV strategies by player A.

Let y_1, y_2, y_3, y_4 are the probabilities of using I, II, III, IV strategies by player B. Solving by using the formula we get,

$$x_1 = \frac{6-3}{4} = \frac{3}{4}, \qquad x_2 = \frac{1}{4}$$

$$x_3 = x_4 = 0, \qquad (\because x_1 + x_2 + x_3 + x_4 = 1)$$

$$y_2 = \frac{6-5}{4} = \frac{1}{4}, \qquad y_4 = \frac{3}{4}$$

$$y_1 = y_3 = 0, \qquad (\because y_1 + y_2 + y_3 + y_4 = 1)$$

$$v = \frac{6.6 - 5.3}{(6+6)-(5+3)}$$

$$= \frac{36-15}{12-8} = \frac{21}{4}$$

28. ***Solve the following game by graphical method, whose pay off matrix to 'A' is given,***

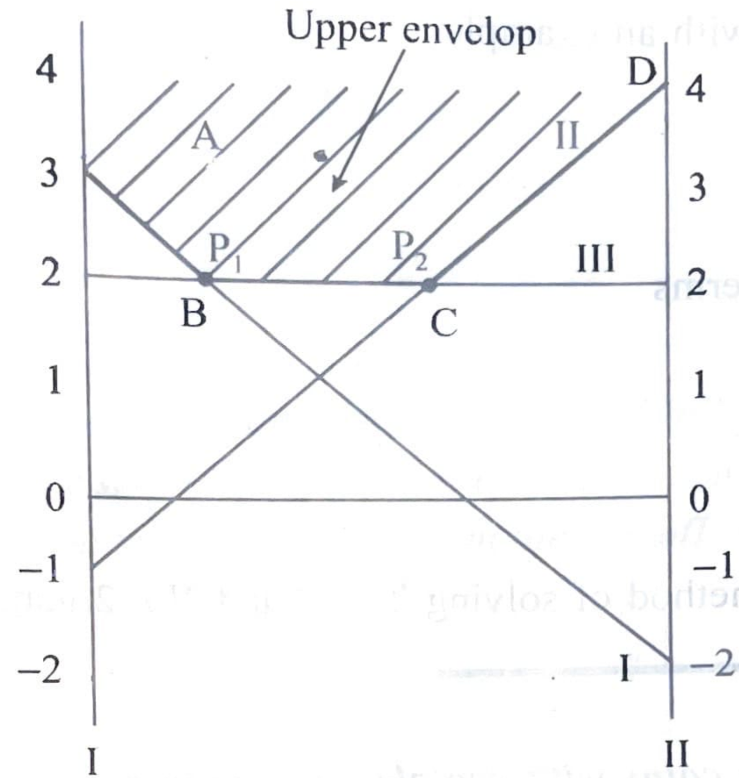
		B		
		I	*II*	*III*
	I	3	2	4
A	*II*	-1	4	2
	III	2	2	6

Solution:

To use graphical method the game should be either 2 × M or N × 2 type. By inspection column I dominates the column III. [As all the elements of column I ≤ column III].

Hence, the reduced matrix is

		B	
		I	II
	I	3	−2
A	II	−1	4
	III	2	2



ABCD is the upper boundary, the lowest point in the upper boundary is same/ repeating at B, C. Hence, the reduced game is

		I	II
A	I	3	−2
	III	2	2

or

		I	II
A	II	−1	4
	III	2	2

The game is having alternative optimal solution, solving the first reduced game by using the formula, the optimal strategies and probabilities are A (I, II, III) = (4/5, 0, 1/5) and for B (I, II, III) = (4/5, 1/5, 0). The value of game $v = 2$.