Operation Research

(15ME81) – CBCS Scheme

Linear Programming Problem Solution using Graphical Method

Module-1 (Topic -3)

By

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LPP Graphical Method:

- The maximization or minimization of some quantity is the objective in all linear programming problems.
- A feasible solution satisfies all the problem's constraints.
- Changes to the objective function coefficients do not affect the feasibility of the problem.
- An optimal solution is a feasible solution that results in the largest possible objective function value, z, when maximizing or smallest z when minimizing.
- In the graphical method, if the objective function line is parallel to a boundary constraint in the direction of optimization, there are alternate optimal solutions, with all points on this line segment being optimal.
- A graphical solution method can be used to solve a linear program with two variables.
- If a linear program possesses an optimal solution, then an extreme point will be optimal.
- If a constraint can be removed without affecting the shape of the feasible region, the constraint is said to be redundant.
- A nonbinding constraint is one in which there is positive slack or surplus when evaluated at the optimal solution.
- A linear program which is over constrained so that no point satisfies all the constraints is said to be infeasible.

Example 1:

Question - Find the graphical solution to the following,

$$\text{Max} \quad Z = 5x_1 + 7x_2$$

$$\text{Sub to,} \quad x_1 \le 6 \text{ ----(1)}$$

$$2x_1 + 3x_2 \le 19 \text{ -----(2)}$$

$$x_1 + x_2 \le 8 \text{ -----(3)}$$

$$x_1, x_2 \ge 0$$

Solution:

Step 1- Replace the inequality sign by equality sign and re-write

x1 = 6 - (1)

2x1 + 3x2 = 19 - (2)

x1 + x2 = 8 - (3)

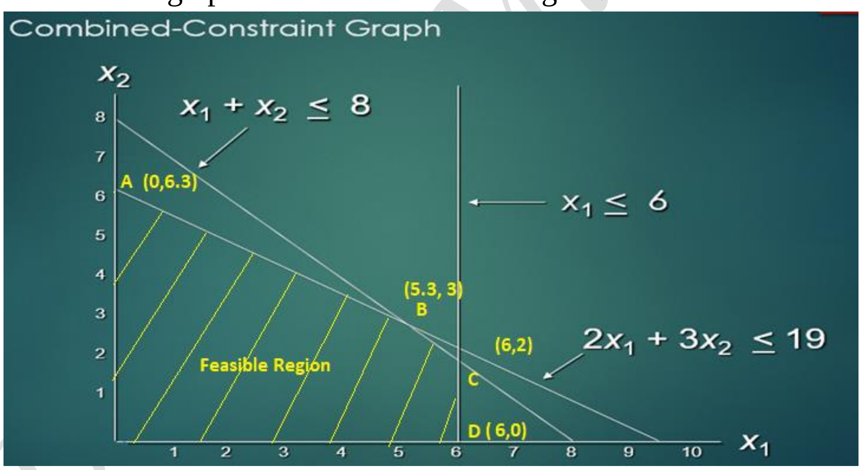
Step 2- Substitute one variable as zero and find another variable value alternatively.

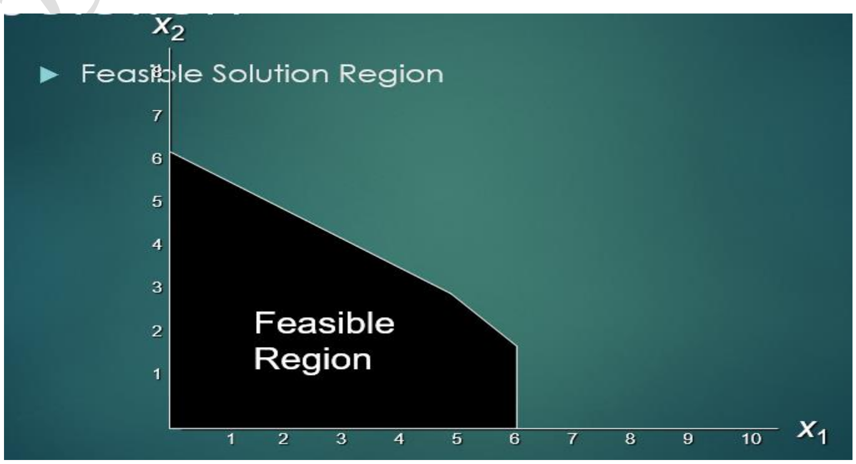
Eqn (1) x1 = 6 and x2 =0

Eqn (2) When x1 = 0 than x2 = 6.5 / When x2 = 0 than x1 = 9.5

Eqn (3) When x1 = 0 than x2 = 8 / When x2 = 0 than x1 = 8

Step 3- Plot these values on the graph and find the feasible region.





Profit at various points:

Location	Coordinates	Profit $Z = 5x_1 + 7x_2$	Remarks
A	(0,6)	42	
B	(5.3,3)	47.5	Maximum
C	(6,2)	44	
D	(6,0)	30	

Answer: X1 = 5.3, X2 = 3   Z_maximum = 47.5

Example 2:

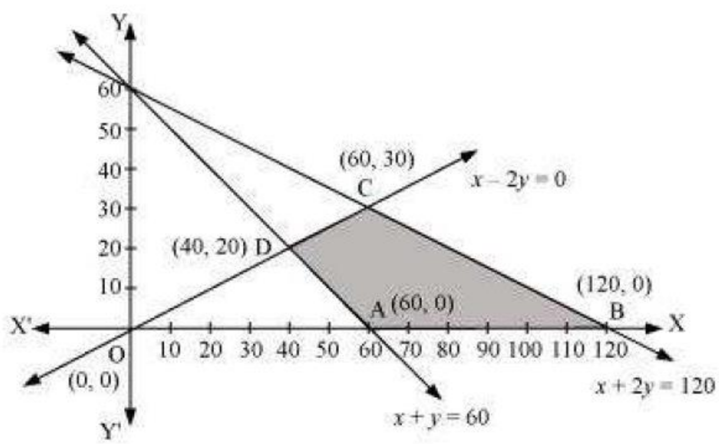
Question :

Minimise and Maximise Z = 5x + 10y

subject to $x+2y \le 120, x+y \ge 60, x-2y \ge 0, x, y \ge 0$.

Answer

The feasible region determined by the constraints, $x + 2y \le 120$, $x + y \ge 60$, $x - 2y \ge 0$, $x \ge 0$, and $y \ge 0$, is as follows.



The corner points of the feasible region are A (60, 0), B (120, 0), C (60, 30), and D (40, 20).

The values of Z at these corner points are as follows.

Corner point	$Z = 5x + 10y$	
A(60, 0)	300	→ Minimum
B(120, 0)	600	→ Maximum
C(60, 30)	600	→ Maximum
D(40, 20)	400	

The minimum value of Z is 300 at (60, 0) and the maximum value of Z is 600 at all the points on the line segment joining (120, 0) and (60, 30).

Example 3:

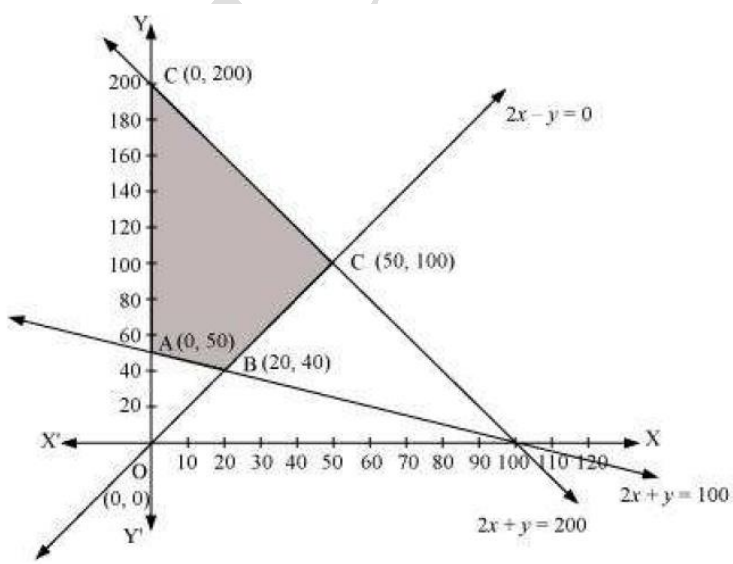
Question :

Minimise and Maximise $Z = x + 2y$

subject to $x + 2y \geq 100,\ 2x - y \leq 0,\ 2x + y \leq 200;\ x, y \geq 0$.

Answer

The feasible region determined by the constraints, $x + 2y \geq 100$, $2x - y \leq 0$, $2x + y \leq 200$, $x \geq 0$, and $y \geq 0$, is as follows.



The corner points of the feasible region are A(0, 50), B(20, 40), C(50, 100), and D(0, 200).

The values of Z at these corner points are as follows.

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Corner point	Z = x + 2y	
A(0, 50)	100	→ Minimum
B(20, 40)	100	→ Minimum
C(50, 100)	250	
D(0, 200)	400	→ Maximum

The maximum value of Z is 400 at (0, 200) and the minimum value of Z is 100 at all the points on the line segment joining the points (0, 50) and (20, 40).

Example 4:

Question : Solve the following LPP graphically using ISO- profit method.

maximize Z =120 X + 100Y.

Subject to the constraints

$10x + 5y \leq 80$

$6x + 6y \leq 66$

$4x + 8y \geq 24$

$5x + 6y \leq 90$

$x \geq 0, \quad y \geq 0$

Answer:

since x≥0, y≥0, consider only the first quadrant of the plane graph the following straight lines on a graph paper

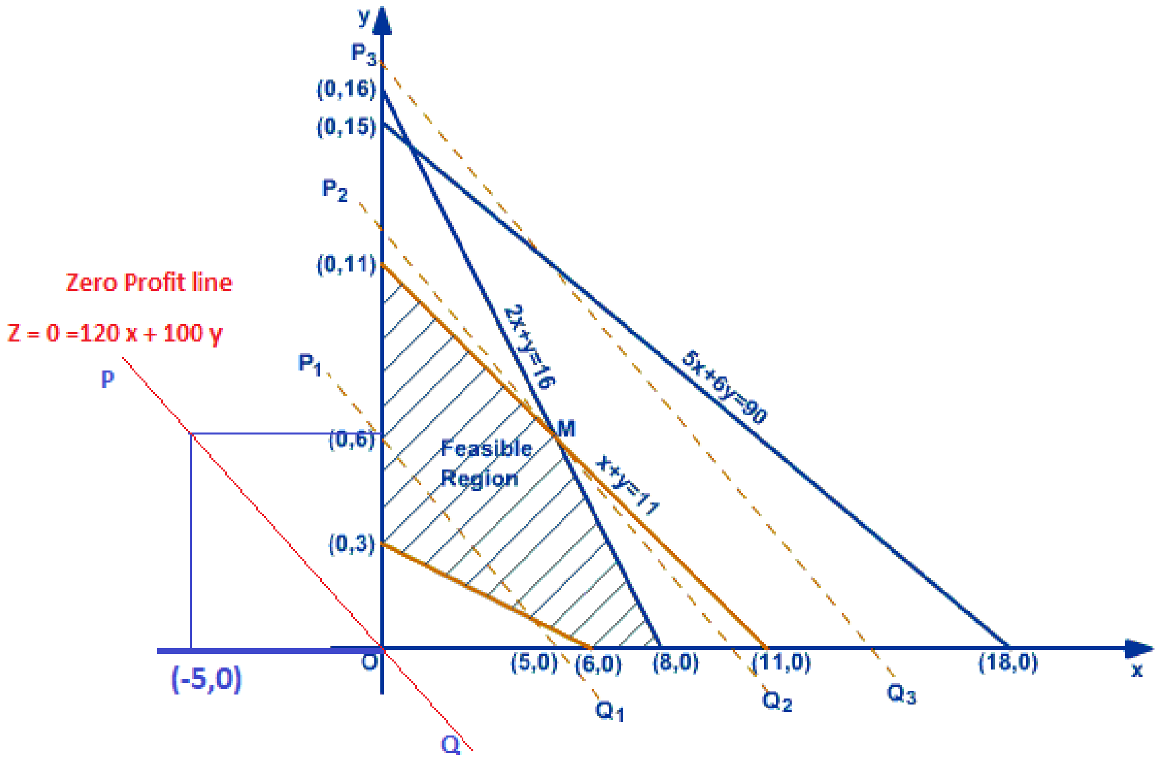
10x + 5y = 80 or 2x+y =16

6x + 6y = 66 or x +y =11

4x+ 8y = 24 or x+ 2y = 6

5x + 6y = 90

Identify all the half planes of the constraints. The intersection of all these half planes is the feasible region as shown in the figure.



Substitute Zero to Z in the objective function, then we have an equation of the line

$120x + 100y = 0$ (1)

or $120x = - 100y$

⇨ $x/y = - 5/6$ So $x = -5$ & $y = 6$

PQ is the Zero profit line,

P_1Q_1, P_2Q_2 and P_3Q_3 are the lines parallel to PQ.

P_2Q_2 is a line parallel to P_1Q_1 and has one point 'M' which belongs to feasible region and farthest from the origin. If we take any line P_3Q_3 parallel to P_2Q_2 away from the origin, it does not touch any point of the feasible region.

The co-ordinates of the point M can be obtained by solving the equation $2x + y = 16$

$x + y =11$ which give $x = 5$ and $y = 6$

⇒The optimal solution for the objective function is **x = 5 and y = 6**

The optimal value of Z 120 (5) + 100 (6) = 600 + 600 = 1200

Special Cases in Graphical Method

Multiple Optimal Solution

Example 5:

Solve by using graphical method

Max Z = $4x_1 + 3x_2$

Subject to $4x_1 + 3x_2 \leq 24$ ---(1)

$x_1 \leq 4.5$ ---(2)

$x_2 \leq 6$ ---(3)

$x_1 \geq 0$, $x_2 \geq 0$

Solution

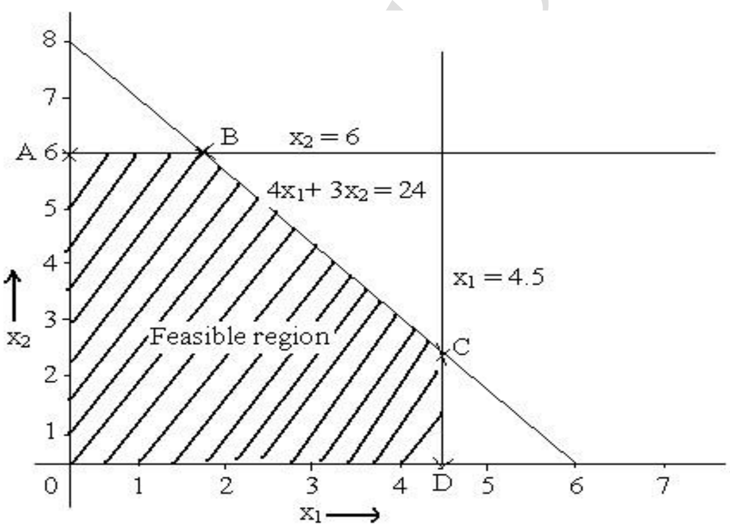
Eqn (1) - $4x1 + 3x2 \leq 24$, written in a form of equation $4x1 + 3x2 = 24$

Put x1 =0, then x2 = 8

Put x2 =0, then x1 = 6 The coordinates are (0, 8) and (6, 0)

Eqn (2) - $x1 \leq 4.5$, written in a form of equation x1=4.5

Eqn (3) - $x2 \leq 6$, written in a form of equation x2 = 6



The corner points of feasible region are A, B, C and D.

1. So the coordinates for the corner points are (0, 6) (1.5, 6)

(Solve the two equations 4x1+ 3x2 = 24 and x2 = 6 to get the coordinates)

2. C (4.5, 2)

(Solve the two equations 4x1+ 3x2 = 24 and x1 = 4.5 to get the coordinates)

3. D (4.5, 0)

We know that Max Z = 4x1 + 3x2

At A (0, 6), Z = 4(0) + 3(6) = 18

At B (1.5, 6), Z = 4(1.5) + 3(6) = 24

At C (4.5, 2), Z = 4(4.5) + 3(2) = 24

At D (4.5, 0), Z = 4(4.5) + 3(0) = 18

Max Z = 24, which is achieved at both B and C corner points. It can be achieved not only at B and C but every point between B and C. Hence the given problem has multiple optimal solutions.

<u>No Optimal Solution</u>

Example 6:

Solve graphically

Max Z = 3x1 + 2x2

Subject to x1+ x2 ≤ 1 x1+ x2 ≥ 3

x1 ≥ 0 , x2 ≥ 0

Solution

The first constraint x1+ x2 ≤ 1, written in a form of equation

x1+ x2 = 1

Put x1 =0, then x2 = 1

Put x2 =0, then x1 = 1

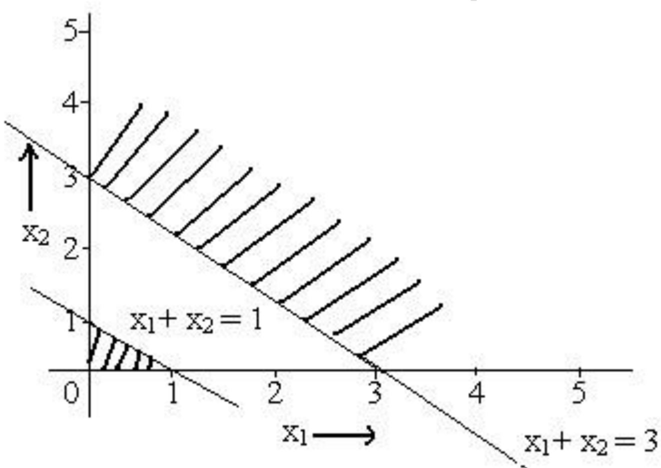
The coordinates are (0, 1) and (1, 0)

The second constraint x1+ x2 ≥ 3, written in a form of equation x1+ x2 = 3

Put x1 =0, then x2 = 3

Put x2 =0, then x1 = 3

The coordinates are (0, 3) and (3, 0)



There is no common feasible region generated by two constraints together i.e. we cannot identify even a single point satisfying the constraints. Hence there is no optimal solution.

Unbounded Solution

Example 7:

Solve by graphical method

Max Z = 3x1 + 5x2

Subject to 2x1+ x2 ≥ 7

x1+ x2 ≥ 6

x1+ 3x2 ≥ 9

x1 ≥ 0 , x2 ≥ 0

Solution

The first constraint 2x1+ x2 ≥ 7, written in a form of equation 2x1+ x2 = 7

Put x1 =0, then x2 = 7

Put x2 =0, then x1 = 3.5

The coordinates are (0, 7) and (3.5, 0)

The second constraint x1+ x2 ≥ 6, written in a form of equation x1+ x2 = 6

Put x1 =0, then x2 = 6

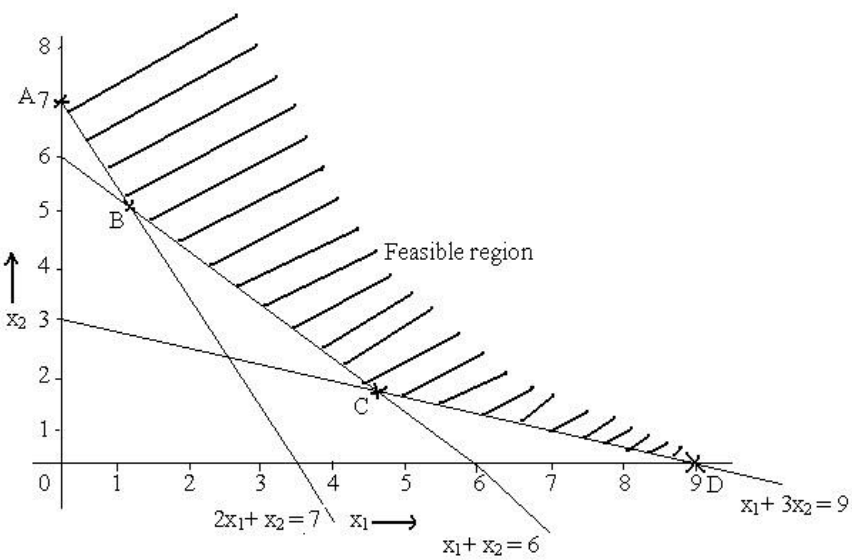
Put x2 =0, then x1 = 6 The coordinates are (0, 6) and (6, 0)

The third constraint x1+ 3x2 ≥ 9, written in a form of equation x1+ 3x2 = 9

Put x1 =0, then x2 = 3

Put x2 =0, then x1 = 9

The coordinates are (0, 3) and (9, 0)



The corner points of feasible region are A, B, C and D. So the coordinates for the corner points are

1. (0, 7)
2. (1, 5) (Solve the two equations $2x1+ x2 = 7$ and $x1+ x2 = 6$ to get the coordinates)
3. C (4.5, 1.5) (Solve the two equations $x1+ x2 = 6$ and $x1+ 3x2 = 9$ to get the coordinates)
4. D (9, 0)

We know that Max $Z = 3x1 + 5x2$
At A (0, 7) $Z = 3(0) + 5(7) = 35$
At B (1, 5) $Z = 3(1) + 5(5) = 28$
At C (4.5, 1.5) $Z = 3(4.5) + 5(1.5) = 21$
At D (9, 0) $Z = 3(9) + 5(0) = 27$
The values of objective function at corner points are 35, 28, 21 and 27. But there exists infinite number of points in the feasible region which is unbounded. The value of objective function will be more than the value of these four corner points i.e. the maximum value of the objective function occurs at a point at ∞. Hence the given problem has unbounded solution.